1. $y = f(x) = \dfrac{3 - e^x}{1 - 3e^x} \iff y - 3ye^x = 3 - e^x$

$\iff y - 3 = 3ye^x - e^x \iff y - 3 = e^x(3y - 1)$

$\iff e^x = \dfrac{y-3}{3y-1} \iff f^{-1}(y) = x = \ln\left(\dfrac{y-3}{3y-1}\right)$

$D_f = V_{f^{-1}}: \; 1 - 3e^x \neq 0 \iff 3e^x \neq 1 \iff e^x \neq \frac{1}{3}$

$\iff x \neq \ln\frac{1}{3} \implies D_f = V_{f^{-1}} = \mathbb{R} \setminus \{\ln\frac{1}{3}\}$

$V_f = D_{f^{-1}}: \; \dfrac{y-3}{3y-1} > 0$

		1/3		3	
$y-3$	−	−	−	0	+
$3y-1$	−	0	+	+	+
$\frac{y-3}{3y-1}$	+	∄	−	0	+

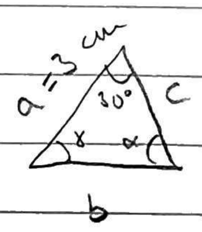
$V_f = D_{f^{-1}} = (-\infty, -\frac{1}{3}) \cup (3, \infty)$

Svar: $f^{-1}(x) = \ln(x-3) - \ln(3y-1)$

$D_{f^{-1}} = (-\infty, -\frac{1}{3}) \cup (3, \infty)$

$V_{f^{-1}} = \mathbb{R} \setminus \{\ln\frac{1}{3}\}$

2.



Area $= 3\sqrt{3}$ cm^2

Areasatsen: $3\sqrt{3} = \dfrac{a \cdot c \cdot \sin\beta}{2} = \dfrac{3 \cdot c \cdot \sin 30^\circ}{2}$

$= \dfrac{3 \cdot c \cdot \frac{1}{2}}{2} = \dfrac{3c}{4} \iff c = 4\sqrt{3}$

Cosinussatsen: $b^2 = 3^2 + (4\sqrt{3})^2 - 2 \cdot 3 \cdot 4\sqrt{3} \cos 30^\circ$

$= 9 + 16 \cdot 3 - 2 \cdot 12\sqrt{3} \cdot \dfrac{\sqrt{3}}{2} = 9 + 48 - 12 \cdot 3 = 9 + 48 - 36 = 21$

$\underset{b>0}{\iff} b = \sqrt{21}$

Sinussatsen: $\frac{\sin\alpha}{3} = \frac{\sin 30°}{\sqrt{21}}$

$\Leftrightarrow \sin\alpha = \frac{3\cdot\sin 30°}{\sqrt{3}\sqrt{7}} = \frac{\sqrt{3}\cdot\frac{1}{2}}{\sqrt{7}} = \frac{\sqrt{3}}{2\sqrt{7}}$

Svar: $\frac{\sqrt{3}}{2\sqrt{7}}$

3a) $4e^{2x} + 4e^{x} = 3$ Sätt $e^x = t$

$4t^2 + 4t - 3 = 0$

$t^2 + t - \frac{3}{4} = 0$

$t = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + \frac{3}{4}} = -\frac{1}{2} \pm \sqrt{1} = -\frac{1}{2} \pm 1$

$t_1 = -\frac{3}{2}$ $t_2 = \frac{1}{2}$

Sätt tillbaka $e^x = t$:

$e^x = t_1$ går ej, ty $e^x > 0$

$e^x = t_2 = \frac{1}{2} \Leftrightarrow x = \ln\frac{1}{2}$

Svar: $x = \ln\frac{1}{2}$

b) $2\cos\left(2x + \frac{\pi}{3}\right)\sin\left(2x - \frac{\pi}{3}\right) = \cos\left(2x + \frac{\pi}{3}\right)$

$\Leftrightarrow 2\cos\left(2x + \frac{\pi}{3}\right)\sin\left(2x - \frac{\pi}{3}\right) - \cos\left(2x + \frac{\pi}{3}\right) = 0$

$\Leftrightarrow \left(2\sin\left(2x - \frac{\pi}{3}\right) - 1\right)\cos\left(2x + \frac{\pi}{3}\right) = 0$

$\Rightarrow 2\sin\left(2x - \frac{\pi}{3}\right) - 1 = 0$ eller $\cos\left(2x + \frac{\pi}{3}\right) = 0$

(1) (2)

Fall 1: $2\sin\left(2x-\frac{\pi}{3}\right)=1 \iff \sin\left(2x-\frac{\pi}{3}\right)=\frac{1}{2}$

$$\iff 2x-\frac{\pi}{3}=\begin{cases}\frac{\pi}{6}+2\pi n \\ \pi-\frac{\pi}{6}+2\pi n\end{cases} \quad n\in\mathbb{Z}$$

$$\iff 2x=\begin{cases}\frac{\pi}{6}+\frac{2\pi}{6}+2\pi n \\ \pi-\frac{\pi}{6}+\frac{2\pi}{6}+2\pi n\end{cases}$$

$$\iff 2x=\begin{cases}\frac{\pi}{2}+2\pi n \\ \frac{7\pi}{6}+2\pi n\end{cases} \iff x=\begin{cases}\frac{\pi}{4}+\pi n \\ \frac{7\pi}{12}+\pi n\end{cases}$$

Fall 2: $2x+\frac{\pi}{3}=\pm\frac{\pi}{2}+2\pi n \iff 2x=\pm\frac{3\pi}{6}-\frac{2\pi}{6}+2\pi n$

$$\iff 2x=\begin{cases}\frac{\pi}{6}+2\pi n \\ -\frac{5\pi}{6}+2\pi n\end{cases} \iff x=\begin{cases}\frac{\pi}{12}+\pi n \\ -\frac{5\pi}{12}+\pi n\end{cases} \iff x=\begin{cases}\frac{\pi}{12}+\pi n \\ \frac{7\pi}{12}+\pi n\end{cases}$$

Svar: $x=\begin{cases}\frac{\pi}{4}+\pi n \\ \frac{7\pi}{12}+\pi n \\ \frac{\pi}{12}+\pi n\end{cases}$ eller $x=\begin{cases}\frac{\pi}{4}+\pi n \\ \frac{\pi}{12}+\frac{\pi}{2}n\end{cases}$ $n\in\mathbb{Z}$

4a) $z_1=1+i\sqrt{3}$ $\quad z_2=1+i$

$$z_3=\frac{z_1}{z_2}=\frac{1+i\sqrt{3}}{1+i}=\frac{(1+i\sqrt{3})(1-i)}{(1+i)(1-i)}=$$

$$=\frac{1-i+i\sqrt{3}+\sqrt{3}}{1+1}=\frac{1+\sqrt{3}+i(\sqrt{3}-1)}{2}$$

Svar: $\frac{1+\sqrt{3}}{2}+i\frac{\sqrt{3}-1}{2}$

4b) $z_1 = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2e^{\frac{\pi}{3}i}$

$z_2 = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\, e^{\frac{\pi}{4}i}$

$$\frac{z_1}{z_2} = \frac{2e^{\frac{\pi}{3}i}}{\sqrt{2}e^{\frac{\pi}{4}i}} = \sqrt{2}\, e^{\left(\frac{\pi}{3} - \frac{\pi}{4}\right)i} = \sqrt{2}\, e^{\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)i}$$

$$= \sqrt{2}\, e^{\frac{\pi}{12}i} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

c) $\begin{cases} \sqrt{2}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2} \\ \sqrt{2}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2} \end{cases}$ enl (a) och (b)

$$\Rightarrow \begin{cases} \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \\ \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \end{cases}$$

Svar: $\begin{cases} \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \\ \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \end{cases}$

5a) $z^5 - 3z^4 + 8z^3 - 14z^2 + 16z - 8 = 0$

$z = 1+i$ rot $\Leftrightarrow$ $z = 1-i$ rot

(komplexkonjugerade rötter)

$\Leftrightarrow$ $z-1-i$ och $z-1+i$ faktorer

$\Leftrightarrow$ $(z-1-i)(z-1+i) = (z-1)^2 - i^2 = z^2 - 2z + 2$

är en faktor

Möjliga heltalsrötter: $\pm 1, \pm 2, \pm 4, \pm 8$

$1^5 - 3 \cdot 1^4 + 8 \cdot 1^3 - 14 \cdot 1^2 + 16 \cdot 1 - 8 = 0 \Rightarrow z = 1$ rot

$\Leftrightarrow$ $z - 1$ faktor

$\Rightarrow (z^2 - 2z + 2)(z-1) = z^3 - z^2 - 2z^2 + 2z + 2z - 2$

$= z^3 - 3z^2 + 4z - 2$ faktor

Polynomdivision:

$$
\begin{array}{l|l}
\quad z^2 + 4 & \\
\hline
z^5 - 3z^4 + 8z^3 - 14z^2 + 16z - 8 & z^3 - 3z^2 + 4z - 2 \\
\underline{-(z^5 - 3z^4 + 4z^3 - 2z^2)} & \\
\qquad 4z^3 - 12z^2 + 16z - 8 & \\
\qquad \underline{-(4z^3 - 12z^2 + 16z - 8)} & \\
\qquad \qquad 0 &
\end{array}
$$

$z^2 + 4$ faktor

$z^2 + 4 = 0 \Leftrightarrow z^2 = -4 \quad z = \pm 2i$

Svar: $z = 1 \pm i$, $z = 1$ och $z = \pm 2i$

b) $(z-2)^4 = (-\sqrt{3}+i)^4$

$-\sqrt{3}+i = 2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = 2(\cos\gamma + i\sin\gamma) = 2e^{\gamma i}$

$\begin{cases} \cos\gamma = -\frac{\sqrt{3}}{2} \\ \sin\gamma = \frac{1}{2} \end{cases}$ 2 kvadranten

$\cos\gamma = -\frac{3}{\sqrt{2}} \Leftrightarrow \cos(\pi-\gamma) = \frac{\sqrt{3}}{2}$

$\sin\gamma = \frac{1}{2} \Leftrightarrow \sin(\pi-\gamma) = \frac{1}{2}$

$\Rightarrow \pi - \gamma = \frac{\pi}{6} \quad \Leftrightarrow \quad \gamma = \frac{5\pi}{6}$

$HL = (-\sqrt{3}+i)^4 = \left(2e^{\frac{5\pi}{6}i}\right)^4 = 2^4 e^{\frac{10\pi}{3}i} = 2^4 e^{\frac{4\pi}{3}i + 2\pi n}$

Låt $z-2 = re^{vi} \Rightarrow VL = (re^{vi})^4 = r^4 e^{4vi}$

Så $r^4 e^{4vi} = 2^4 e^{\frac{4\pi}{3}i + 2\pi n}$

$\Rightarrow r = 2$

$4v = \frac{4\pi}{3} + 2\pi n \Leftrightarrow v = \frac{4\pi}{3} + \frac{\pi}{2}n \quad n \in \mathbb{Z}$

$v_1 = \frac{4\pi}{3} \qquad v_2 = \frac{4\pi}{3} + \frac{\pi}{2} = \frac{8\pi}{6} + \frac{3\pi}{6} = \frac{11\pi}{6}$

$v_3 = \frac{4\pi}{3} + \pi = \frac{7\pi}{3} \qquad v_4 = \frac{4\pi}{3} + \frac{3\pi}{2} = \frac{8\pi}{6} + \frac{9\pi}{6} = \frac{17\pi}{6}$

$$z-2=2e^{\frac{4\pi}{3}i}=2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$

$$=2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-1-i\sqrt{3} \iff z=1-i\sqrt{3}$$

$$z-2=2e^{\frac{11\pi}{6}i}=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

$$=2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\sqrt{3}-i \iff z=2+\sqrt{3}-i$$

$$z-2=2e^{\frac{7\pi}{3}i}=2\left(\cos\frac{7\pi}{3}+i\sin\frac{7\pi}{3}\right)$$

$$=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=1+i\sqrt{3} \iff z=3+i\sqrt{3}$$

$$z-2=2e^{\frac{17\pi}{6}i}=2\left(\cos\frac{17\pi}{6}+i\sin\frac{17\pi}{6}\right)$$

$$=2\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=-\sqrt{3}+i \iff z=2-\sqrt{3}+i$$

Svar: $z=1-i\sqrt{3}$, $z=3+i\sqrt{3}$, $z=2+\sqrt{3}-i$, $z=2-\sqrt{3}+i$

6a) $$\lim_{x\to-\infty}\left(\ln(8x^2-3x)-\ln(2x^2+1)\right)$$

$$=\lim_{x\to-\infty}\ln\left(\frac{8x^2-3x}{2x^2+1}\right)=\lim_{x\to-\infty}\ln\left(\frac{x^2(8-3/x)}{x^2(2+1/x^2)}\right)$$

$$=\lim_{x\to-\infty}\ln\left(\frac{8-3/x}{2+1/x^2}\right)=\ln\frac{8}{2}=\ln 4$$

Svar: $\ln 4$

b) $$\lim_{x\to 3}\frac{x^2-x-6}{x\sin(x-3)}=\lim_{x\to3}\frac{(x-3)(x+2)}{x\sin(x-3)}$$

$$=\underbrace{\lim_{x\to3}\frac{x-3}{\sin(x-3)}}_{\lim_{x\to0}\frac{x}{\sin x}=1}\cdot\lim_{x\to3}\frac{x+2}{x}=\lim_{x\to3}\frac{x+2}{x}=\frac{5}{3}$$

(standardgränsvärde, enl klämsatsen)

Svar: $\frac{5}{3}$

c) $$\lim_{x\to\frac{\pi}{2}}\frac{\cos(x-\pi)}{\ln\left(\sqrt{x+e^2-\frac{\pi}{2}}\right)\tan\left(x-\frac{\pi}{2}\right)}$$

$$=\left[\cos(x-\pi)=\cos(\pi-x)=\sin\left(\frac{\pi}{2}-\pi+x\right)=\sin\left(x-\frac{\pi}{2}\right)\right]$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{\sin\left(x-\frac{\pi}{2}\right)}{\ln\left(\sqrt{x+e^2-\frac{\pi}{2}}\right)\cdot\frac{\sin\left(x-\frac{\pi}{2}\right)}{\cos\left(x-\frac{\pi}{2}\right)}}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\cos\left(x-\frac{\pi}{2}\right)}{\ln\left(\sqrt{x+e^2-\frac{\pi}{2}}\right)}=\frac{\cos 0}{\ln(e^1)}=\frac{1}{1}=1$$

Svar: 1

7 $f(x) = \begin{cases} \dfrac{ax+b}{x^2+2x-3} & x<1 \\ 3 & x=1 \\ \dfrac{\sqrt{x+a^2-1}-a}{x-1} & x>1 \end{cases}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$ ska gälla

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{\sqrt{x+a^2-1}-a}{x-1} = \left[\text{förläng med konjugatet}\right]$

$= \lim_{x\to 1^+} \dfrac{x+a^2-1-a^2}{(x-1)(\sqrt{x+a^2-1}+a)} = \lim_{x\to 1^+} \dfrac{x-1}{(x-1)(\sqrt{x+a^2-1}+a)}$

$= \lim_{x\to 1^+} \dfrac{1}{\sqrt{x+a^2-1}+a} = \dfrac{1}{|a|+a} = \dfrac{1}{2a}$ om $a>0$

Obs! Om $a<0$ blir nämnaren 0

$f(1)=3$ så $\dfrac{1}{2a}=3 \Leftrightarrow a=\dfrac{1}{6}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \dfrac{ax+b}{x^2+2x-3} = \lim_{x\to 1^-} \dfrac{ax+b}{(x+3)(x-1)}$

$= \lim_{x\to 1^-} \dfrac{\frac{1}{6}x-b}{(x+3)(x-1)} = \lim_{x\to 1^-} \dfrac{\frac{1}{6}(x-6b)}{(x+3)(x-1)}$

existerar om "$\frac{0}{0}$" så $x-6b = x-1$

$\Leftrightarrow 6b=1 \Leftrightarrow b=\frac{1}{6}$

$\lim_{x\to 1^-} \dfrac{\frac{1}{6}(x-1)}{(x+3)(x-1)} = \lim_{x\to 1^-} \dfrac{1}{6(x+3)} = \dfrac{1}{24}$

Så $\lim_{x\to 1^-} f(x) = \dfrac{1}{24} \neq 3 = f(1)$.

Alltså kan funktionen inte bli kontinuerlig. ■

8. $\log_a x = y \iff a^{\log_a x} = a^y$

$\iff x = a^y \iff \ln x = \ln a^y$

$\iff \ln x = y \cdot \ln a \iff y = \frac{\ln x}{\ln a}$

Svar: $\log_a x = \frac{\ln x}{\ln a}$