MVE426, Matematik del B, Tekniskt basår

Lösningar tentamen 14/1 – 23

1(a) $2\ln(xy) + \ln\left(\frac{1}{x}+\frac{1}{y}\right) - \ln(x^3+y^3) + \ln(x^2-xy+y^2) =$

$$= \ln(x^2y^2) + \ln\left(\frac{y+x}{xy}\right) + \ln\left(\frac{x^2-xy+y^2}{x^3+y^3}\right) =$$

$$= \ln\left(x^2y^2\cdot\frac{y+x}{xy}\right) + \ln\left(\frac{x^2-xy+y^2}{(x+y)(x^2-xy+y^2)}\right) =$$

$$= \ln(xy(x+y)) + \ln\left(\frac{1}{x+y}\right) = \ln\left(xy(x+y)\cdot\frac{1}{x+y}\right) = \underline{\underline{\ln(xy)}}$$

(b) $x = f^{-1}(y) \iff y = f(x) = 1 + \ln(\sqrt{x-2}) \iff$

$$\iff \ln(\sqrt{x-2}) = y-1 \iff \sqrt{x-2} = e^{y-1} \implies$$

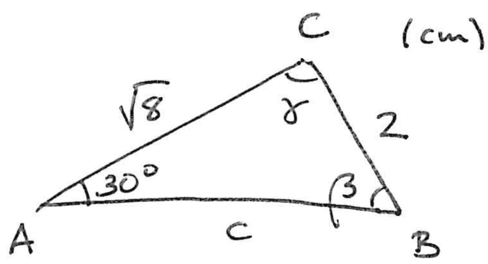
$$\implies x-2 = e^{2(y-1)}$$

$$\therefore\ x = f^{-1}(y) = \underline{\underline{2+e^{2(y-1)}}}$$

$$D_{f^{-1}} = V_f = \underline{\underline{\mathbb{R}}}$$

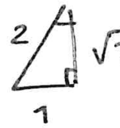
$$V_{f^{-1}} = D_f = \underline{\underline{(2,\infty)}}$$

2. Sinussatsen:



$$\frac{\sin(30^\circ)}{2} = \frac{\sin(\beta)}{\sqrt{8}} \Leftrightarrow$$

$$\Leftrightarrow \sin(\beta) = \frac{\sqrt{8}\sin(30^\circ)}{2} =$$

$$= \left\{ \right.$$  $$\left. \right\} = \frac{2\sqrt{2}\cdot\frac{1}{2}}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \beta = \arcsin\left(\frac{1}{\sqrt{2}}\right) = 45^\circ \text{ eller } \beta = 180^\circ - \arcsin\left(\frac{1}{\sqrt{2}}\right) = 135^\circ$$

<u>Fall 1, $\beta = 45^\circ$:</u> $\gamma = 180^\circ - 30^\circ - 45^\circ = 105^\circ$

Sinussatsen: $\frac{\sin(\gamma)}{c} = \frac{\sin(30^\circ)}{2} \Leftrightarrow$

$$\Leftrightarrow c = \frac{2\sin(\gamma)}{\sin(30^\circ)} = \frac{2\sin(105^\circ)}{\frac{1}{2}} = 4\sin(105^\circ) =$$

$$= 4\sin(60^\circ + 45^\circ) = 4\left(\sin(60^\circ)\cos(45^\circ) + \sin(45^\circ)\cos(60^\circ)\right) =$$

$$= 4\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right) = \sqrt{6} + \sqrt{2}$$

<u>Fall II, $\beta = 135^\circ$:</u> $\gamma = 180^\circ - 30^\circ - 135^\circ = 15^\circ$

Sinussatsen: $\frac{\sin(\gamma)}{c} = \frac{\sin(30^\circ)}{2} \Leftrightarrow c = \frac{2\sin(\gamma)}{\sin(30^\circ)} =$

$$= 4\sin(15^\circ) = 4\sin(45^\circ - 30^\circ) =$$

$$= 4\left(\sin(45^\circ)\cos(30^\circ) - \sin(30^\circ)\sin(45^\circ)\right) =$$

$$= 4\left(\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right) = \sqrt{6} - \sqrt{2}$$

$\therefore$ I. $\alpha = 30^\circ$, $\beta = 45^\circ$, $\gamma = 105^\circ$, $a = 2$, $b = \sqrt{8}$, $c = \sqrt{6} + \sqrt{2}$

II. $\alpha = 30^\circ$, $\beta = 135^\circ$, $\gamma = 15^\circ$, $a = 2$, $b = \sqrt{8}$, $c = \sqrt{6} - \sqrt{2}$

3. (a) $\lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{27x^3+13}}{\sqrt{5x^2+7x+2}} = \lim\limits_{x\to-\infty} \dfrac{\sqrt[3]{x^3}\,\sqrt[3]{27+\frac{13}{x^3}}}{\sqrt{x^2}\sqrt{5+\frac{7}{x}+\frac{2}{x^2}}} =$

$= \left\{ \begin{array}{l} \sqrt{x^2} = |x| = -x \text{ då} \\ x\to-\infty \text{ ger } x<0 \end{array} \right\} = \lim\limits_{x\to-\infty} \dfrac{\cancel{x}\sqrt[3]{27+\frac{13}{x^3}}}{-\cancel{x}\sqrt{5+\frac{7}{x}+\frac{2}{x^2}}} =$

$= \dfrac{\sqrt[3]{27}}{-\sqrt{5}} = \underline{\underline{-\dfrac{3}{\sqrt{5}}}}$

(b) $\dfrac{\sqrt{x^2-x+1}-\sqrt{x^2+x+1}}{\tan(2x)} \cdot \dfrac{\sqrt{x^2-x+1}+\sqrt{x^2+x+1}}{\sqrt{x^2-x+1}+\sqrt{x^2+x+1}} =$

$= \dfrac{x^2-x+1-(x^2+x+1)}{\frac{\sin(2x)}{\cos(2x)}\cdot\left(\sqrt{x^2-x+1}+\sqrt{x^2+x+1}\right)} =$

$= \dfrac{-\cancel{2x}^{\,1}}{\frac{\sin(2x)}{2x}\cdot\frac{\cancel{2x}}{1}\cdot\frac{1}{\cos(2x)}\cdot\left(\sqrt{x^2-x+1}+\sqrt{x^2+x+1}\right)} \xrightarrow[x\to 0]{}$

$\xrightarrow[x\to 0]{} \dfrac{-1}{1\cdot\frac{1}{1}\cdot\left(\sqrt{1}+\sqrt{1}\right)} = \underline{\underline{-\dfrac{1}{2}}}$

4. (a) $\ln(x+2) - \ln(2-x) = 3\ln(2) \Rightarrow$

$$\Rightarrow \ln\left(\frac{x+2}{2-x}\right) = \ln(2^3) \Leftrightarrow \frac{x+2}{2-x} = 8 \Leftrightarrow$$

$$\Leftrightarrow x+2 = 8(2-x) \Leftrightarrow x+2 = 16-8x$$

$$\Leftrightarrow 9x = 14 \Leftrightarrow x = \frac{14}{9}$$

Test: $VL = \ln\left(\frac{14}{9}+2\right) - \ln\left(2-\frac{14}{9}\right) =$

$$= \ln\left(\frac{32}{9}\right) - \ln\left(\frac{4}{9}\right) = \ln\left(\frac{32}{9}\Big/\frac{4}{9}\right) =$$

$$= \ln\left(\frac{32}{4}\right) = \ln(8) = \ln(2^3) = 3\ln(2) = HL \text{ ok!}$$

$$\therefore \; x = \frac{14}{9}$$

(b) $3\tan(x) = 2\cos(x) \Leftrightarrow \dfrac{3\sin(x)}{\cos(x)} = 2\cos(x)$

$$\Leftrightarrow 3\sin(x) = 2\cos^2(x) \Leftrightarrow \{\cos^2(x) = 1-\sin^2(x)\}$$

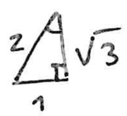
$$\Leftrightarrow 3\sin(x) = 2(1-\sin^2(x)) \Leftrightarrow$$

$$\Leftrightarrow 2\sin^2(x) + 3\sin(x) - 2 = 0$$

Låt $t = \sin(x)$: $2t^2+3t-2=0 \Leftrightarrow t^2+\frac{3}{2}t-1=0$

$$t = -\frac{3}{4} \pm \sqrt{\frac{9}{16}+1} = -\frac{3}{4} \pm \frac{5}{4} \Rightarrow \begin{array}{l} t_1 = -2 \\ t_2 = \frac{1}{2} \end{array}$$

I. $\sin(x) = -2$ ↯ Går ej då $-1 \leq \sin(x) \leq 1$.

II. $\sin(x) = \frac{1}{2}$, $\arcsin\left(\frac{1}{2}\right) = \{$ 

$\} = \frac{\pi}{6}$

$$\Rightarrow x = \begin{cases} \frac{\pi}{6} + 2\pi n \\ \text{eller} \\ \pi - \frac{\pi}{6} + 2\pi n \end{cases} \Leftrightarrow x = \begin{cases} \frac{\pi}{6} + 2\pi n \\ \text{eller} \\ \frac{5\pi}{6} + 2\pi n \end{cases} \quad n \in \mathbb{Z}$$

$$\therefore \; x = \frac{\pi}{6} + 2\pi n \;\text{ eller }\; x = \frac{5\pi}{6} + 2\pi n \;, \quad n \in \mathbb{Z}$$

(c) $\sqrt{3}\sin(x) + \cos(x) = \sqrt{3}$

Vi använder omskrivn. $a \cdot \cos(v) + b \cdot \sin(v) = c\sin(v + \phi)$

I vårt fall: $a = 1$, $b = \sqrt{3}$, $v = x$

$$c = \sqrt{a^2 + b^2} = \sqrt{1+3} = 2$$

$$b = \sqrt{3} > 0 \Rightarrow \phi = \arctan\left(\frac{a}{b}\right) =$$

$$= \arctan\left(\frac{1}{\sqrt{3}}\right) = \Bigg\{$$  $$\Bigg\} = \frac{\pi}{6}$$

$$\Rightarrow \cos(x) + \sqrt{3}\sin(x) = \sqrt{3} \Leftrightarrow 2\sin\left(x + \frac{\pi}{6}\right) = \sqrt{3}$$

$$\Leftrightarrow \sin\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \Rightarrow$$

$$\Rightarrow x + \frac{\pi}{6} = \begin{cases} \frac{\pi}{3} + 2\pi n \\ \text{eller} \\ \pi - \frac{\pi}{3} + 2\pi n \end{cases} \Leftrightarrow x = \begin{cases} \frac{\pi}{3} - \frac{\pi}{6} + 2\pi n \\ \text{eller} \\ \frac{2\pi}{3} - \frac{\pi}{6} + 2\pi n \end{cases} \quad n \in \mathbb{Z}$$

$$\therefore \; x = \frac{\pi}{6} + 2\pi n \;\text{ eller }\; x = \frac{\pi}{2} + 2\pi n \;, \quad n \in \mathbb{Z}$$

5. (a) $\frac{z}{2i} - \frac{\bar{z}}{3i} = \frac{5}{3} - \frac{1}{6}i$. mult. båda sidor med $6i$

$$3z - 2\bar{z} = 10i - i^2$$

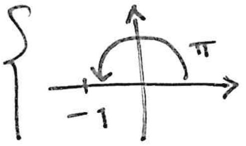
Låt $z = x + iy$: $3(x+iy) - 2(x-iy) = 1 + 10i$

$$\Leftrightarrow 3x + 3yi - 2x + 2yi = 1 + 10i \Leftrightarrow$$

$$\Leftrightarrow x + 5y \cdot i = 1 + 10 \cdot i$$

$$\Rightarrow \begin{cases} x = 1 \\ 5y = 10 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$$

$$\therefore \underline{\underline{z = 1 + 2i}}$$

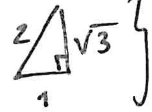
(b) $-1 = \Bigg\{$  $\Bigg\} = 1e^{i\pi} = e^{i(\pi + 2\pi k)} \quad k \in \mathbb{Z}$

Detta ger: $z^6 = -1 \Leftrightarrow z^6 = e^{i(\pi + 2\pi k)}$, $k \in \mathbb{Z}$

Om $z = re^{i\theta}$ blir detta: $(re^{i\theta})^6 = e^{i(\pi + 2\pi k)}$

$$\Leftrightarrow r^6 e^{i \cdot 6\theta} = 1 \cdot e^{i(\pi + 2\pi k)} \quad k \in \mathbb{Z}$$

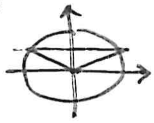
$$\Rightarrow \begin{cases} r^6 = 1 \\ 6\theta = \pi + 2\pi k \end{cases} \Leftrightarrow \begin{cases} r = 1 \\ \theta = \frac{\pi}{6} + \frac{\pi}{3} \cdot k \end{cases} \quad k \in \mathbb{Z}$$

$\underline{k=0}$: $z_1 = 1 \cdot e^{i\frac{\pi}{6}} = \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}) = \Big\{$  $\Big\} =$

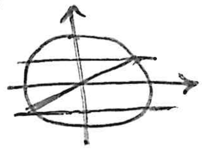
$$= \frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}$$

$\underline{k=1}$: $z_2 = 1 \cdot e^{i\left(\frac{\pi}{6}+\frac{\pi}{3}\right)} = e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

$\underline{k=2}$: $z_3 = 1 \cdot e^{i\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)} = e^{i\frac{5\pi}{6}} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) =$

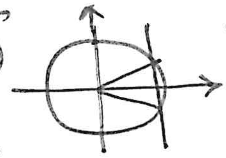
$= \Big\{$  $\Big\}$ $= -\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}$

$\underline{k=3}$: $z_4 = 1 \cdot e^{i\left(\frac{\pi}{6}+\pi\right)} = e^{i\frac{7\pi}{6}} = \cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right) =$

$= \Big\{$  $\Big\}$ $= -\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} - i \cdot \frac{1}{2}$

$\underline{k=4}$: $z_5 = 1 \cdot e^{i\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)} = e^{i\frac{3\pi}{2}} = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$

$\underline{k=5}$: $z_6 = 1 \cdot e^{i\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)} = e^{i\frac{11\pi}{6}} = \cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right) =$

$= \Big\{$  $\Big\}$ $= \cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - i \cdot \frac{1}{2}$

$\therefore z_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $z_2 = i$, $z_3 = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$,

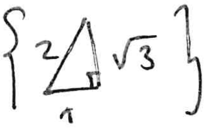
$z_4 = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$, $z_5 = -i$, $z_6 = \frac{\sqrt{3}}{2} - i\frac{1}{2}$

6. (a) Låt $z_1 = -\sqrt{3} + i$, $z_2 = 2 - \sqrt{12} \cdot i$ och $z_3 = 1 - i$.

Vill skriva om dessa på polär form.

$$|z_1| = \sqrt{3+1} = 2$$

$$\arg(z_1) \overset{x<0}{=} \arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi = \pi - \arctan\left(\frac{1}{\sqrt{3}}\right) =$$

$= \{$ 

$\} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$$|z_2| = \sqrt{4 + 12} = 4$$

$$\arg(z_2) \overset{x>0}{=} \arctan\left(\frac{-\sqrt{12}}{2}\right) = -\arctan\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$|z_3| = \sqrt{1+1} = \sqrt{2}$$

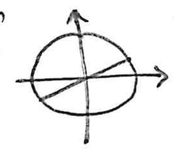
$$\arg(z_3) \overset{x>0}{=} \arctan\left(\frac{-1}{1}\right) = -\arctan(1) = -\frac{\pi}{4}$$

$$\Rightarrow \frac{(-\sqrt{3}+i)^7 (2-\sqrt{12}\cdot i)^5}{(1-i)^{28}} = \frac{(2e^{i\frac{5\pi}{6}})^7 \cdot (4e^{-i\frac{\pi}{3}})^5}{(\sqrt{2}e^{-i\frac{\pi}{4}})^{28}} =$$

$$= \frac{2^7 \cdot e^{i\frac{35\pi}{6}} \cdot 4^5 e^{-i\frac{5\pi}{3}}}{2^{14} \cdot e^{-i\cdot 7\pi}} = \frac{(2^2)^5}{2^7} \cdot e^{i\left(\frac{35\pi}{6} - \frac{5\pi\cdot 2}{3\cdot 2} + \cancel{6\pi} + \pi\right)} =$$

$$= 2^3 \cdot e^{i\left(\frac{25\pi}{6} + \frac{6\pi}{6}\right)} = 8e^{i\frac{31\pi}{6}} = 8e^{i\cdot\frac{24\pi + 7\pi}{6}} =$$

$$= 8e^{i\cdot\left(\cancel{4\pi} + \frac{7\pi}{6}\right)} = 8\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) =$$

$= \{$  $\} = 8\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right) = 8\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) =$

$$= -4\sqrt{3} - 4i = -4(\sqrt{3} + i)$$

(b) $$\frac{1}{1-3i}+\frac{1}{1+ai}=\frac{1}{1-3i}\cdot\frac{1+3i}{1+3i}+\frac{1}{1+ai}\cdot\frac{1-ai}{1-ai}=$$

$$=\frac{1+3i}{1-9i^2}+\frac{1-ai}{1-a^2\cdot i^2}=\frac{1+3i}{10}+\frac{1-ai}{1+a^2}=$$

$$=\frac{1+3i}{10}\cdot\frac{1+a^2}{1+a^2}+\frac{1-ai}{1+a^2}\cdot\frac{10}{10}=$$

$$=\frac{(1+3i)(1+a^2)+10-10ai}{10(1+a^2)}=$$

$$=\frac{1+a^2+3i+3a^2\cdot i+10-10ai}{10(1+a^2)}=$$

$$=\frac{11+a^2+3a^2\cdot i-10ai+3i}{10(1+a^2)}=$$

$$=\frac{11+a^2}{10(1+a^2)}+i\cdot\frac{3a^2-10a+3}{10(1+a^2)}$$

Reellt då $\mathrm{Im}=0$, vilket sker om

$$3a^2-10a+3=0 \iff a^2-\frac{10}{3}a+1=0$$

$$a=\frac{5}{3}\pm\sqrt{\frac{25}{9}-1}=\frac{5}{3}\pm\frac{4}{3}$$

$\therefore$ $a=3$ eller $a=\frac{1}{3}$

7. (a) <u>Definition</u>: En funktion $f(x)$ sägs vara <u>kontinuerlig</u> i en punkt $a \in D_f$ om

$$\lim_{x \to a} f(x) = f(a).$$

(b) Alla tre uttrycken är väldef. och kont. på sina resp. intervall, så för att $f(x)$ ska bli kont. $\forall x \in \mathbb{R}$ krävs att $f(x)$ är kont. i $x = 0$ och $x = 4$.

$\underline{x=0}$: $\displaystyle \lim_{x \to 0^-} f(x) = \left\{ \begin{array}{l} x \to 0^- \\ \Rightarrow x < 0 \end{array} \right\} = \lim_{x \to 0^-} \frac{\tan(6x)}{2x} =$

$$= \lim_{x \to 0^-} \frac{\sin(6x)}{\cos(6x)} \cdot \frac{1}{2x} = \lim_{x \to 0^-} \frac{\sin(6x)}{6x} \cdot \frac{6\cancel{x}}{1} \cdot \frac{1}{2\cancel{x}\cos(6x)} =$$

$$= 1 \cdot \frac{6}{1} \cdot \frac{1}{2 \cdot 1} = 3$$

$$\lim_{x \to 0^+} f(x) = \left\{ \begin{array}{l} x \to 0^+ \\ \Rightarrow x > 0 \end{array} \right\} = \lim_{x \to 0^+} (a\sqrt{x} + b) = b = f(0)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \Leftrightarrow \underline{\underline{3 = b}}$$

$\underline{x=4}$: $\displaystyle \lim_{x \to 4^-} f(x) = \left\{ \begin{array}{l} x \to 4^- \\ \Rightarrow x < 4 \end{array} \right\} = \lim_{x \to 4^-} (a\sqrt{x} + b) = 2a + b = f(4)$

$$\lim_{x \to 4^+} f(x) = \left\{ \begin{array}{l} x \to 4^+ \\ \Rightarrow x > 4 \end{array} \right\} = \lim_{x \to 4^+} \frac{x^2 - 3x - 4}{x^2 - 6x + 8} =$$

$$= \left\{ \begin{array}{l} x^2 - 3x - 4 = 0 \Rightarrow x = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 4} = \frac{3}{2} \pm \frac{5}{2} \Rightarrow \begin{array}{l} x_1 = 4 \\ x_2 = -1 \end{array} \\ \\ x^2 - 6x + 8 = 0 \Rightarrow x = 3 \pm \sqrt{9 - 8} = 3 \pm 1 \Rightarrow \begin{array}{l} x_1 = 4 \\ x_2 = 2 \end{array} \end{array} \right\} =$$

$$= \lim_{x \to 4^+} \frac{\cancel{(x-4)}(x+1)}{\cancel{(x-4)}(x-2)} = \lim_{x \to 4^+} \frac{x+1}{x-2} = \frac{5}{2}$$

$$f(4) = \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) \iff 2a+b = \frac{5}{2}$$

$$\iff \left\{ \begin{array}{c} \text{Vet att} \\ b=3 \end{array} \right\} \iff 2a+3 = \frac{5}{2} \iff 2a = -\frac{1}{2}$$

$\therefore$ $f(x)$ kont. $\forall x \in \mathbb{R}$ om $a = -\frac{1}{4}$ och $b=3$