MVE426, Matematik del A, Tekniskt basår

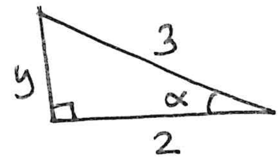
Lösningar, tenta kl. 14.00 – 18.00, 27/10 – 20

1 (a)
$$\frac{\frac{b^2}{a+b}+a-b}{1-\frac{(a-b)^2}{(a+b)^2}} = \frac{\frac{b^2+(a-b)(a+b)}{a+b}}{\frac{(a+b)^2-(a-b)^2}{(a+b)^2}} =$$

$$= \frac{\frac{b^2+a^2-b^2}{a+b}}{\frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{(a+b)^2}} = \frac{a^2}{a+b} \Big/ \frac{4ab}{(a+b)^2} =$$

$$= \frac{a^2}{a+b}\cdot\frac{(a+b)^2}{4ab} = \underline{\underline{\frac{a(a+b)}{4b}}}$$

(b) $\alpha = \arccos\left(\frac{2}{3}\right) \iff \cos(\alpha) = \frac{2}{3}$



Pyth. sats: $2^2+y^2=3^2 \iff y^2 = 9-4 = 5$

$$\overset{y\geq 0}{\Longrightarrow} \quad y=\sqrt{5}$$

$\sin(\alpha) = \frac{y}{3} = \frac{\sqrt{5}}{3}$, $\tan(\alpha) = \frac{y}{2} = \frac{\sqrt{5}}{2}$

2 (b) Pyth. sats:

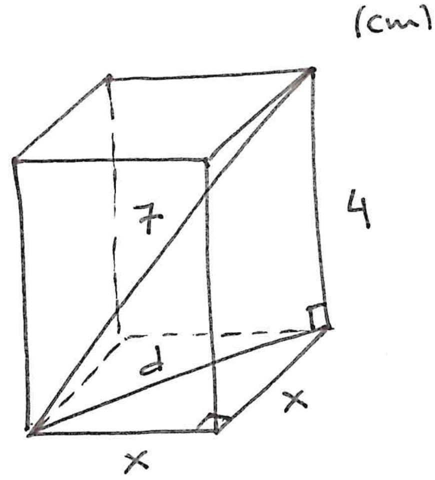
$$x^2 + x^2 = d^2 \Leftrightarrow 2x^2 = d^2$$

Pyth. sats (igen!):

$$d^2 + 4^2 = 7^2 \Leftrightarrow 2x^2 + 16 = 49$$

$$\Leftrightarrow 2x^2 = 33 \Leftrightarrow x^2 = \frac{33}{2}$$

$$\text{Volym} = x^2 \cdot 4 = \frac{33}{2} \cdot 4 = \underline{\underline{66 \text{ cm}^3}}$$



3. $\begin{cases} 2x - 4y + 2z = 2 & \boxed{\tfrac{1}{2}} \\ 3x + y + z = -1 \\ x - y + z = 3 \end{cases} \Longleftrightarrow \begin{cases} x - 2y + z = 1 & \boxed{-3}\ \boxed{-1} \\ 3x + y + z = -1 \\ x - y + z = 3 \end{cases} \Longleftrightarrow$

$\Longleftrightarrow \begin{cases} x - 2y + z = 1 \\ 7y - 2z = -4 \\ y = 2 \end{cases} \Longleftrightarrow \begin{cases} x - 2y + z = 1 \\ y = 2 & \boxed{-7} \\ 7y - 2z = -4 \end{cases} \Longleftrightarrow$

$\Longleftrightarrow \begin{cases} x - 2y + z = 1 \\ y = 2 \\ -2z = -18 & \boxed{-\tfrac{1}{2}} \end{cases} \Longleftrightarrow \begin{cases} x - 2y + z = 1 \\ y = 2 \\ z = 9 & \boxed{-1} \end{cases} \Longleftrightarrow$

$\Longleftrightarrow \begin{cases} x - 2y = -8 \\ y = 2 & \boxed{2} \\ z = 9 \end{cases} \Longleftrightarrow \begin{cases} x = -4 \\ y = 2 \\ z = 9 \end{cases}$

$\therefore\ (x, y, z) = (-4, 2, 9)$

4(a) $$\frac{x-4}{x+2} \leq \frac{x^2+2}{x^2-1} \iff \frac{x-4}{x+2} - \frac{x^2+2}{(x-1)(x+1)} \leq 0 \iff$$

$$\iff \frac{(x-4)(x^2-1)-(x^2+2)(x+2)}{(x+2)(x-1)(x+1)} \leq 0 \iff$$

$$\iff \frac{x^3-x-4x^2+4-(x^3+2x^2+2x+4)}{(x-1)(x+1)(x+2)} \leq 0 \iff$$

$$\iff \frac{-6x^2-3x}{(x-1)(x+1)(x+2)} \leq 0 \iff \frac{-3x(2x+1)}{(x-1)(x+1)(x+2)} \leq 0$$

$$\iff \frac{x(2x+1)}{(x-1)(x+1)(x+2)} \geq 0 \iff \frac{2x(x+\frac{1}{2})}{(x-1)(x+1)(x+2)} \geq 0$$

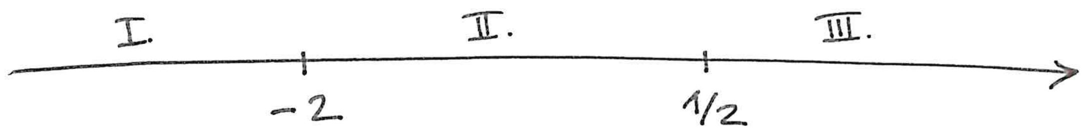
$$\iff R(x) = \frac{\overset{0}{\overset{\wr}{x}}\,(\overset{-1/2}{\overset{\wr}{x+\tfrac{1}{2}}})}{(\underset{\underset{1}{\wr}}{x-1})(\underset{\underset{-1}{\wr}}{x+1})(\underset{\underset{-2}{\wr}}{x+2})} \geq 0$$

		-2		-1		$-1/2$		0		1	
$x+2$	$-$	0	$+$		$+$		$+$		$+$		$+$
$x+1$	$-$		$-$	0	$+$		$+$		$+$		$+$
$x+\frac{1}{2}$	$-$		$-$		$-$	0	$+$		$+$		$+$
x	$-$		$-$		$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$		$-$		$-$	0	$+$
$R(x)$	$-$	↯	$+$	↯	$-$	0	$+$	0	$-$	↯	$+$

$$\therefore\ x \in (-2,-1) \cup \left[-\tfrac{1}{2}, 0\right] \cup (1,\infty)$$

4(b) Vill rita kurvan: $y = |2x-1| - 3|x+2|$ $(*)$

Brytpunkter: $x = -2$, $x = \frac{1}{2}$



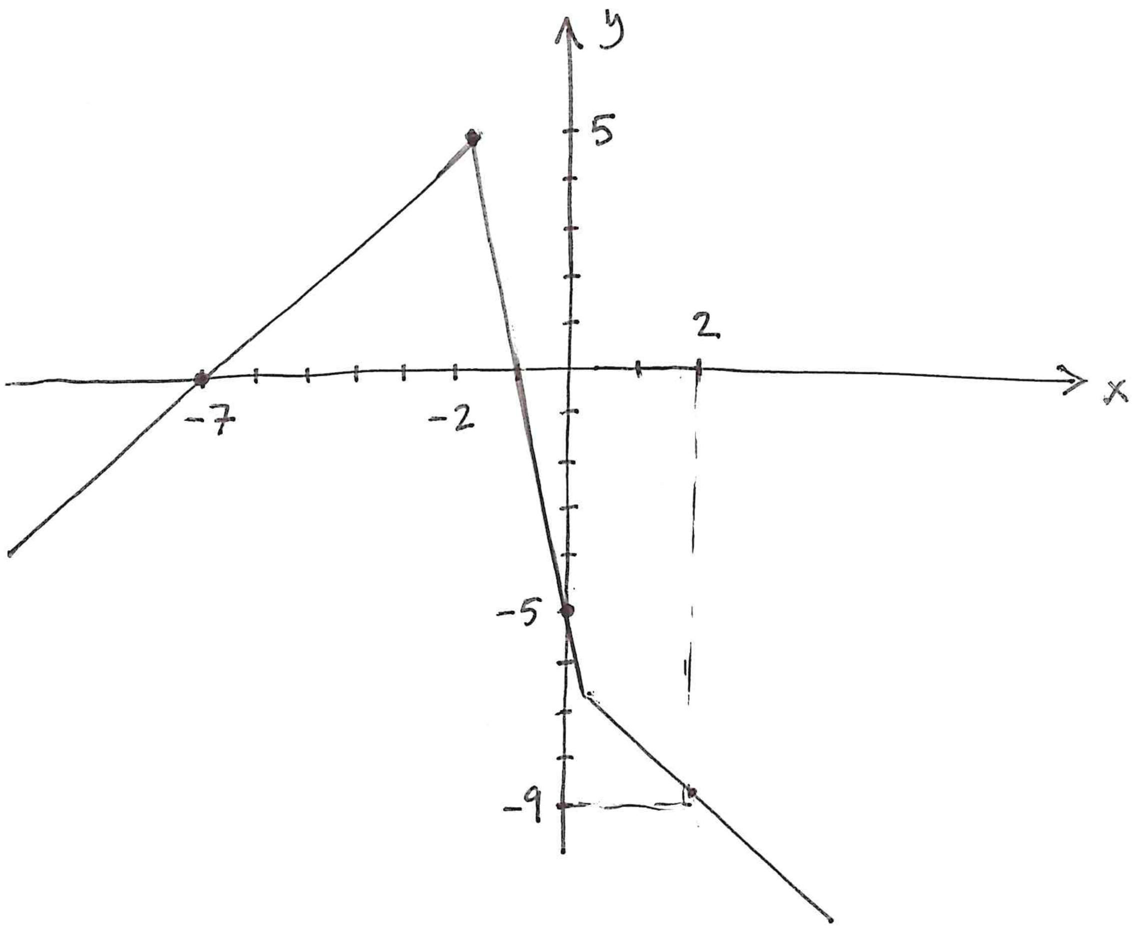
I. $\underline{x < -2}$: $(*) \Leftrightarrow y = -(2x-1) - 3(-(x+2)) \Leftrightarrow$

$\Leftrightarrow y = x + 7$

II. $\underline{-2 \leq x < 1/2}$: $(*) \Leftrightarrow y = -(2x-1) - 3(x+2) \Leftrightarrow$

$\Leftrightarrow y = -5x - 5$

III. $\underline{\frac{1}{2} \leq x}$: $(*) \Leftrightarrow y = 2x - 1 - 3(x+2) \Leftrightarrow y = -x - 7$



Antal lösningar till $|2x-1|-3|x+2|=C$

$$\Longleftrightarrow$$

Antal skärningspunkter mellan:

$y=|2x-1|-3|x+2|$ och $y=C$

Vi ser att det finns:

2 lösningar då $C\in(-\infty,5)$

1 lösning då $C=5$

Inga lösningar då $C\in(5,\infty)$

5. $$\frac{x}{x^2-4x+4} - \frac{3x-4}{x^3-5x^2+8x-4} \qquad (*)$$

Vi vill faktorisera nämnarna:

$$x^2-4x+4 = x^2-2\cdot x\cdot 2+2^2-2^2+4 = \qquad (**)$$
$$= (x-2)^2-4+4 = (x-2)^2$$

Låt $P(x) = x^3-5x^2+8x-4$. $\quad a_0 = -4 = -1\cdot 2\cdot 2$

$\Longrightarrow$ Möjliga heltalsrötter: $\pm 1, \pm 2, \pm 4$ (enligt satsen om heltalsrötter)

$$P(-1) = -1-5-8-4 \neq 0$$
$$P(1) = 1-5+8-4 = 0 \Longrightarrow x_1 = 1$$

$$\begin{array}{r|l} & x^2-4x+4 \\ \hline x-1 & x^3-5x^2+8x-4 \\ & \underline{-(x^3-x^2)} \\ & \quad -4x^2+8x-4 \\ & \quad \underline{-(-4x^2+4x)} \\ & \qquad 4x-4 \\ & \qquad \underline{-(4x-4)} \leftarrow \text{ok!} \\ & \qquad 0 \end{array}$$

$$\therefore P(x) = (x-1)(x^2-4x+4) \overset{(**)}{=} (x-1)(x-2)^2$$

Detta ger:

$$(*) = \frac{x}{(x-2)^2} - \frac{3x-4}{(x-1)(x-2)^2} =$$

$$= \frac{x(x-1)-(3x-4)}{(x-1)(x-2)^2} = \frac{x^2-x-3x+4}{(x-1)(x-2)^2} =$$

$$= \frac{x^2 - 4x + 4}{(x-1)(x-2)^2} \overset{(**)}{=} \frac{(x-2)^2}{(x-1)(x-2)^2} = \frac{1}{x-1}$$

$$\therefore \frac{x}{x^2-4x+4} - \frac{3x-4}{x^3-5x^2+8x-4} = \underline{\underline{\frac{1}{x-1}}}$$

6. $$\frac{x^2}{k^2-16} + \frac{y^2}{5+4k} = 1$$

(a) Cirkel om $k^2-16=5+4k$ och nämnarna är >0.

$$k^2-16=5+4k \iff k^2-4k-21=0$$

$$k = 2 \pm \sqrt{4+21} = 2 \pm 5$$

Om $k_1 = 7$ så $5+4k_1 = 5+28 > 0$ ok!

Om $k_2 = -3$ så $5+4k_2 = 5-12 < 0$ falsk rot!

$\therefore$ Cirkel om $k=7$

(b) Cirkel om $k^2-16>0$ och $5+4k>0$

$k^2-16>0 \iff k^2>16$ så $k \in (-\infty,-4) \cup (4,\infty)$

$5+4k>0 \iff 4k>-5$ så $k \in (-\frac{5}{4}, \infty)$

$\therefore$ Ellips om $k \in (4,\infty)$