Modelling and Simulation ESS101
21 August 2023, Re-exam

This exam contains 11 pages (including this cover page) and 5 problems.

You are allowed to use the following material:

- *Modelling And Simulation, Lecture notes for the Chalmers course ESS101*, by S. Gros (with brief annotations, but cannot contain solutions to the exercises or previous exams)
- *Mathematics Handbook* (Beta)
- *Physics Handbook*
- Chalmers approved calculator
- Formula sheet, appended to the exam.

– Organize your work in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering may receive less credit.

– Mysterious or unsupported answers will not receive credit, but an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

– None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.

– The passing grade will be given at 20 points, grade 4 at 27 and the top grade at 34 points.

Problem	Points	Score
1	8	
2	6	
3	10	
4	7	
5	9	
Total:	40	

Best of luck to all !!

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1. Consider a hanging chain made of N dimensionless balls of mass m (see Fig. 1 for an illustration). The balls are connected to each other by rigid cables of length L, and the two balls at the extremities of the chain are connected to the two points $\mathbf{p}_0(t)$, $\mathbf{p}_{N+1}(t) \in \mathbb{R}^3$ with rigid cables of length L as well.

   (a) (8 points) Write down the model equations of the chain (in 3D) in the form of a semi-explicit index-3 DAE. Note that the end points can move in time!

   Note: try to keep your notations compact. You do not need to provide $\frac{\partial \mathbf{C}}{\partial \mathbf{q}}$ explicitly (where $\mathbf{q}$ will be your set of generalised coordinates), but you need to detail the model enough that one would understand how to code it symbolically in the computer (i.e. using basic operations like Jacobians and matrix-vector multiplications)

Figure 1: Hanging chain with N=10.

Solution:

(a) Let us describe the hanging chain via the generalized coordinates (z-axis up):

$$\mathbf{q} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_N \end{bmatrix} \in \mathbb{R}^{3N} \tag{1}$$

specifying the position of each ball. The chain then has the kinetic and potential energy:

$$\mathcal{T} = \frac{1}{2} m \sum_{i=1}^{N} \dot{\mathbf{p}}_i^\top \dot{\mathbf{p}}_i = \frac{1}{2} m \dot{\mathbf{q}}^\top \dot{\mathbf{q}}, \qquad \mathcal{V} = mg \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \sum_{i=1}^{N} \mathbf{p}_i = mg\mathbf{a}^\top \mathbf{q} \tag{2}$$

where $\mathbf{a}^\top = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & \dots \end{bmatrix}$, and has the constraint functions

$$\mathbf{C}(\mathbf{q}, t) = \begin{bmatrix} C_1 \\ \vdots \\ C_{N+1} \end{bmatrix}, \qquad C_k(\mathbf{q}, t) = \frac{1}{2} \left(\left\| \mathbf{p}_k - \mathbf{p}_{k-1} \right\|^2 - L^2 \right) \tag{3}$$

for $k = 1, \dots, N+1$. The Lagrange function then reads as:

$$\mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}, \mathbf{z}, t) = \frac{1}{2} m \dot{\mathbf{q}}^\top \dot{\mathbf{q}} - mg\mathbf{a}^\top \mathbf{q} - \mathbf{z}^\top \mathbf{C}(\mathbf{q}, t) \tag{4}$$

The Lagrange equation then provides the dynamics, using:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = m\dot{\mathbf{q}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = m\ddot{\mathbf{q}}, \quad \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = -mg\mathbf{a} - \mathbf{z}^\top \frac{\partial \mathbf{C}}{\partial \mathbf{q}} \tag{5}$$

Hence we have the index-3 DAE model:

$$\dot{\mathbf{q}} = \mathbf{v} \tag{6a}$$

$$\dot{\mathbf{v}} = -g\mathbf{a} - \frac{1}{m}\frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \mathbf{z} \tag{6b}$$

$$0 = \mathbf{C}(\mathbf{q}, t) \tag{6c}$$

This result could also be obtained directly from (29) by observing that $W(\mathbf{q}) = I$. We can additionally further explicitly provide (though this is not required from the question):

$$\frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top = \begin{bmatrix} \mathbf{p}_1 - \mathbf{p}_0 & \mathbf{p}_1 - \mathbf{p}_2 & 0 & 0 & \dots & & & \\ 0 & \mathbf{p}_2 - \mathbf{p}_1 & \mathbf{p}_2 - \mathbf{p}_3 & 0 & \dots & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & & \vdots & 0 & \mathbf{p}_{N-1} - \mathbf{p}_{N-2} & \mathbf{p}_{N-1} - \mathbf{p}_N & 0 \\ & & \vdots & 0 & 0 & \mathbf{p}_N - \mathbf{p}_{N-1} & \mathbf{p}_N - \mathbf{p}_{N+1} \end{bmatrix} \tag{7}$$

2. (a) (6 points) Consider the following DAE, where u is the input:

$$\dot{x}_1 = x_1 + x_2 + z \tag{8a}$$

$$\dot{x}_2 = z + u \tag{8b}$$

$$0 = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right) \tag{8c}$$

1. (1 point) Why is it a DAE?
2. (3 points) What is the differential index of (8)?
3. (2 points) What is the DAE obtained by reducing the index of (8) to index=1?

Solution:

1. We can answer in a simple or formal way:
   - The simple way relies on observing that variable z does not enter as time-differentiated in (8), it is an algebraic variable and therefore (8) forms a DAE.
   - For the complex and formal way, we observe that (8) is given by the fully implicit differential equation:

$$F(\dot{\mathbf{s}}, \mathbf{s}, u) = \begin{bmatrix} \dot{x}_1 - x_1 - x_2 - z \\ \dot{x}_2 - z - u \\ \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right) \end{bmatrix} \tag{9}$$

where $\mathbf{s} = \begin{bmatrix} x_1 & x_2 & z \end{bmatrix}^\top$. Then observing that the matrix

$$\frac{\partial F}{\partial \dot{\mathbf{s}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

is rank-deficient, it follows that (8) is a DAE.

2. We observe that (8) is a semi-explicit DAE with

$$g(\mathbf{x}, z, u) = \frac{1}{2}\left(x_1^2 + x_2^2 - 1\right) \tag{11}$$

and $\frac{\partial g}{\partial z} = 0$, hence it is of index larger than 1. In order to compute precisely the differential index, we need to perform time-differentiations on (8) until it is transformed in an ODE. Because (8a)-(8b) are already ODEs (functions of z), we can leave them alone, and focus on (8c). We then observe that:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{x}, z, u) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left(x_1^2 + x_2^2 - 1\right)\right) = x_1\dot{x}_1 + x_2\dot{x}_2 \tag{12}$$

Replacing $\dot{x}_1$, $\dot{x}_2$ by there expressions from (8a)-(8b), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{x}, z, u) = x_1(x_1 + x_2 + z) + x_2(z + u) \tag{13}$$

which is not yet a differential equations. A second time-derivative delivers:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\mathbf{x}, z, u) = \dot{x}_1(x_1 + x_2 + z) + \dot{x}_2(z + u) + x_1(\dot{x}_1 + \dot{x}_2 + \dot{z}) + x_2(\dot{z} + \dot{u}) \tag{14}$$

We can then solve $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\mathbf{x}, z, u) = 0$ for $\dot{z}$:

$$\dot{z} = \frac{-\dot{x}_1(x_1 + x_2 + z) - \dot{x}_2(z + u) - x_1\dot{x}_1 - x_1\dot{x}_2 - x_2\dot{u}}{x_1 + x_2} \tag{15}$$

(and then replacing $\dot{x}_1$ and $\dot{x}_2$ by there expressions from (8a)-(8b) as above) which is an ODE as long as $x_1 + x_2 \neq 0$. Hence, the original DAE has index 2.

3. We have already performed this task in the previous question. The index-reduced DAE is the one occurring “one step before getting an ODE”, i.e. we can write:

$$\dot{x}_1 = x_1 + x_2 + z \tag{16a}$$
$$\dot{x}_2 = z + u \tag{16b}$$
$$0 = x_1(x_1 + x_2 + z) + x_2(z + u) \tag{16c}$$

Here as well we need $x_1 + x_2 \neq 0$ to be able to solve (16c).

3. (a) (3 points) Consider the curve fitting results in Figure 2a,b,c.
   1. (2 points) Comment on which curve fitting is best. How is better fitting determined? Using a least squares approach capturing the relationship between x and y, and the model $\hat{\mathbf{y}} = \boldsymbol{\theta}^\top \boldsymbol{\varphi}$, where $\boldsymbol{\theta}$ is the parameter vector and $\boldsymbol{\varphi}$ is the *regression vector*, discuss the differences between the parameter vectors that lead to fitting results in Figure 2a and Figure 2 b.
   2. (1 points) Comment on the quality of the model fitting in Figure 2c. Is it an acceptable fitting result? What could be potential problems and how they can be resolved?

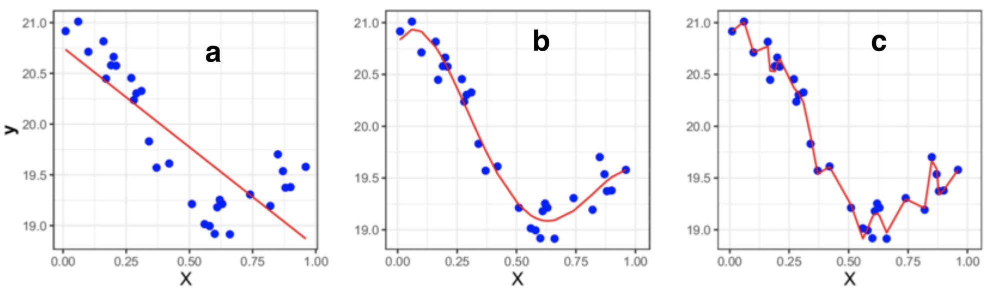

Figure 2: Linear regression examples based on least squares.

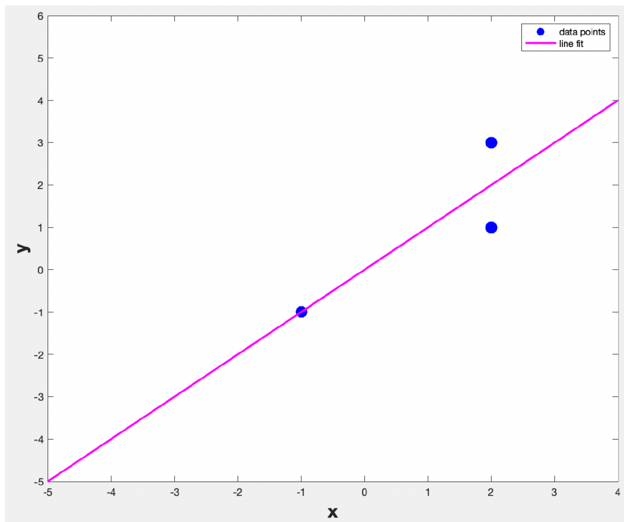

Figure 3: Illustration of three data points and least-squares based model fitting result.

(b) (1 point) What is a loss function, give an example, how is it used in model fitting? How would loss function values change with respect to increasing model order?

(c) (2 points) In Figure 3, the data pairs are given as $\{[x(i), y(i)]\} = \{[2, 1], [2, 3], [-1, -1]\}$. Using a least squares approach capturing the relationship between x and y, $\hat{\mathbf{y}} = \boldsymbol{\theta}^\top \boldsymbol{\varphi}$ (where $\boldsymbol{\varphi}$ is the *regression vector* holding the *regressors* $(1, x, ...)$), find the values in the parameter vector θ that corresponds to the model fit in the figure.

(d) (4 points) Consider the following ARX one-step-ahead predictor

$$\hat{y}(t) = ay(t-1) + bu(t-1). \tag{17}$$

Assume that the following data set is available

$$[y(0),\ y(1)] = [0,\ 1] \tag{18}$$
$$[u(0),\ u(1)] = [1,\ 0] \tag{19}$$

Write the predictor (17) in the linear regression form $\hat{y}(t) = h(t)^T\theta$ and find the least-squares estimate for a, b, given the available data.

Solution:

(a) 1. Model fit in Figure 2b is better, in comparison to Figure 2a, as the fitting error is less.

Model fit quality can be determined by a loss function based on the differences between the predictions and the ground truth values. There are two parameters in line fitting and three in the quadratic model fitting.

2. Figure 2c shows overfitting which is problematic as it leads to bad generalization performance, i.e. model predictions on unseen data would not be of good quality. Cross-validation can help to find a good model order to avoid overfitting.

(b) Loss function is used to evaluate how well a model fits the data. A basic example is mean squared error based on the difference between predictions and the ground truth data. Loss function values decrease as the model fits well to the data, i.e., typically when the model order is incrreased.

(c) Using the data vectors $x = [2, 2, -1]$ and $y = [1, 3, -1]$ in the least-squares estimate where $k = 1$ for fitting a line:

$$\hat{\boldsymbol{\theta}}_N = R_N^{-1} f_N = \big(\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\varphi}(i)\boldsymbol{\varphi}^\top(i)\big)^{-1} \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\varphi}(i) y(i)$$

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\varphi}(i)\boldsymbol{\varphi}^\top(i) = \frac{1}{N}\begin{bmatrix} N & \sum_{i=1}^{N} x(i) & \dots & \sum_{i=1}^{N} x^k(i) \\ \sum_{i=1}^{N} x(i) & \sum_{i=1}^{N} x^2(i) & \dots & \sum_{i=1}^{N} x^{k+1}(i) \\ . & . & . & \\ . & . & . & \\ . & . & . & \\ \sum_{i=1}^{N} x^k(i) & \sum_{i=1}^{N} x^{k+1}(i) & \dots & \sum_{i=1}^{N} x^{2*k}(i) \end{bmatrix} \tag{20}$$

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\varphi}(i) y(i) = \frac{1}{N}\begin{bmatrix} \sum_{i=1}^{N} y(i) \\ \sum_{i=1}^{N} x(i)y(i) \\ . \\ . \\ . \\ \sum_{i=1}^{N} x^k(i)y(i) \end{bmatrix} \tag{21}$$

The resulting parameter estimate $\hat{\boldsymbol{\theta}}_N = [0, 1]^T$, that corresponds to $a = 0$, $b = 1$ in $\hat{\mathbf{y}} = a + bx = \boldsymbol{\theta}^\top \boldsymbol{\varphi}$.

(d) The linear regression form is

$$y(t) = [y(t-1) \quad u(t-1)] \begin{bmatrix} a \\ b \end{bmatrix}, \tag{22}$$

and the least squares estimate of the parameters can be found by $\hat{\theta}_{LS} = (H^T H)^{-1} H^T \mathbf{y}$. Using the available data we have that

$$H = \begin{bmatrix} y(0) & u(0) \\ y(1) & u(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{23}$$

Hence the least squares estimate is $\hat{\theta}_{LS} = [a \quad b] = [1 \quad 0]$.

4. (a) (4 points) Given the following Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

1. (3 points) Write down the corresponding RK method equations.
2. (1 points) Is it an explicit or implicit approach, why? How many stages are there?

(b) (3 points) Consider an integration scheme, described by the following Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

1. (2 points) Determine the stability function.
2. (1 point) Is the scheme A-stable?

Solution:

(a)
- $$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{\Delta t}{4}\mathbf{f}\big(\mathbf{x}_k, \mathbf{u}\,(t_k)\big) + \frac{3\Delta t}{4}\mathbf{f}\Big(\mathbf{x}_k + \frac{2\Delta t}{3}\mathbf{f}\big(\mathbf{x}_k, \mathbf{u}\,(t_k)\big), \mathbf{u}\left(t_k + \frac{2\Delta t}{3}\right)\Big)$$
- Explicit, the tableau is lower triangular, Ks can be computed sequentially, there are 2 stages.

(b)
1. Denoting the Butcher array as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

the stability function is given by $R(\mu) = 1 + \mu b^T(I - \mu A)^{-1}\mathbf{1}$, where $\mu = \lambda\Delta t$ and $\mathbf{1}$ is a column vector with all entries equal to 1. Thus:

$$R(\mu) = 1 + \mu \begin{bmatrix}1/2 & 1/2\end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu/2 & 1-\mu/2 \end{bmatrix}^{-1} \begin{bmatrix}1\\1\end{bmatrix} = \frac{1+\mu/2}{1-\mu/2}$$

2. Since $|1+\mu/2| \leq |1-\mu/2|$ for all μ in the left half-plane, $|R(\mu)| \leq 1$ for the same μ, i.e. the scheme is A-stable.

5. The Newton method aims at solving a set of equations $\mathbf{r}\,(\mathbf{x}) = 0$ numerically. To that end, it iterates the recursion:

$$\frac{\partial \mathbf{r}\,(\mathbf{x})}{\partial \mathbf{x}}\Delta\mathbf{x} + \mathbf{r}\,(\mathbf{x}) = 0 \tag{24a}$$

$$\mathbf{x} \leftarrow \mathbf{x} + \alpha\Delta\mathbf{x} \tag{24b}$$

where $\alpha \in]0,\,1]$ is the step-size.

(a) (3 points) The optimization problem

$$\text{minimize}_{\mathbf{x}}\ \Phi(\mathbf{x}, \mathbf{p}), \tag{25}$$

where $\mathbf{x}$ is the optimization variable and $\mathbf{p}$ is a vector of parameters, can be approached by applying the Newton method.

1. (1 point) How is the function $\mathbf{r}(\mathbf{x})$ chosen in this case?
2. (1 point) Give a sufficient condition for (25) to have a unique solution $\mathbf{x}^\star(\mathbf{p})$ for a given $\mathbf{p}$.
3. (1 point) When could Newton method fail?

(b) (6 points) Consider the function $\mathbf{f} : \mathbb{R}^2 \mapsto \mathbb{R}^2$, by $f(x,y) = \begin{bmatrix} 3x^2 + y^2 - 7 \\ x^2 + 4y^2 - 8 \end{bmatrix}$, for which we construct approximate solutions to the equation $f(x,y) = [0,0]^T$ using Newton method and the initial guess $\mathbf{x} = [x_0, y_0] = [1,1]$. Calculate the resulting solution from applying Newton iteration only once, i.e. $[x_1, y_1]$.

Solution:

(a)
1. The Newton method is applied to the (necessary) condition for a solution:

$$\mathbf{r}(\mathbf{x}) = \nabla_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{p}) = 0 \tag{26}$$

2. Problem (25) is guaranteed to have a unique solution if it is convex, i.e. if the Hessian $\nabla_{\mathbf{xx}} \Phi(\mathbf{x}, \mathbf{p})$ is positive definite.
3. When the Jacobian is singular, i.e. the Newton direction is undefined. This is a common failure mode, when tackling very nonlinear functions and starting with a poor initial guess.
4. $\frac{\partial f(x,y)}{\partial \mathbf{x}} = \begin{bmatrix} 6x & 2y \\ 2x & 8y \end{bmatrix}$, $det \partial f(x,y) = 44xy$, $\partial f(x,y)^{-1} = \frac{1}{44xy} \begin{bmatrix} 8y & -2y \\ -2x & 6x \end{bmatrix} = \begin{bmatrix} \frac{2}{11x} & \frac{-1}{22x} \\ \frac{-1}{22y} & \frac{3}{22y} \end{bmatrix}$

   Thus, $\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \partial f(x_{k-1}, y_{k-1})^{-1} f(x_{k-1}, y_{k-1}) =$

   $\begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \begin{bmatrix} \frac{2}{11x_{k-1}} & \frac{-1}{22x_{k-1}} \\ \frac{-1}{22y_{k-1}} & \frac{3}{22y_{k-1}} \end{bmatrix} \begin{bmatrix} 3x_{k-1}^2 + y_{k-1}^2 - 7 \\ x_{k-1}^2 + 4y_{k-1}^2 - 8 \end{bmatrix}$

   Beginning with $x_0 = y_0 = 1$, using the resulting equations above, we get$x_1 = 1.4$, $y_1 = 1.27$

Appendix: some possibly useful formula

- Lagrange mechanics is built on the equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}, \qquad \mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}\right) = \mathcal{T} - \mathcal{V} - \mathbf{z}^\top \mathbf{C}, \qquad \mathbf{C} = 0, \qquad \langle \delta\mathbf{q}, \mathbf{Q}\rangle = \delta W, \forall\, \delta\mathbf{q} \tag{27}$$

The kinetic and potential energy of a point mass are given by:

$$\mathcal{T} = \frac{1}{2}m\dot{\mathbf{p}}^\top\dot{\mathbf{p}}, \qquad \mathcal{V} = mg\mathbf{p}_3 \tag{28}$$

respectively, where $\mathbf{p} \in \mathbb{R}^3$ is the position of the mass in a cartesian reference frame having the third coordinate as the vertical axis pointing up. The generalized forces are identical to the external forces applied to a point mass if the position of that point is expressed in cartesian coordinates in the generalized coordinates $\mathbf{q}$.

- In the case $\mathcal{T} = \frac{1}{2}m\dot{\mathbf{q}}^\top W\dot{\mathbf{q}}$ with W constant $\mathcal{V} = \mathcal{V}\left(\mathbf{q}\right)$ and $\mathbf{C} = \mathbf{C}\left(\mathbf{q}\right)$, the Lagrange equations simplify to the dynamics in the semi-explicit index-3 DAE form:

$$\dot{\mathbf{p}} = \mathbf{v} \tag{29a}$$

$$W\dot{\mathbf{v}} + \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \mathbf{z} = \mathbf{Q} - \frac{\partial \mathcal{V}}{\partial \mathbf{q}}^\top \tag{29b}$$

$$0 = \mathbf{C}\left(\mathbf{q}\right) \tag{29c}$$

- The Implicit Function Theorem (IFT) guarantees that a nonlinear set of equations

$$\mathbf{r}\left(\mathbf{y}, \mathbf{z}\right) = 0 \tag{30}$$

"can be solved" in terms of $\mathbf{z}$ for a given $\mathbf{y}$ iff the Jacobian $\frac{\partial \mathbf{r}(\mathbf{y},\mathbf{z})}{\partial \mathbf{z}}$ is full rank at the solution. More specifically, it guarantees that there is a function $\phi\left(\mathbf{y}\right)$ such that

$$\mathbf{r}\left(\mathbf{y}, \phi\left(\mathbf{y}\right)\right) = 0 \tag{31}$$

holds in the neighborhood of the point $\mathbf{y}$ where the Jacobian is evaluated. Furthermore, the IFT specifies that:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{r}}{\partial \mathbf{z}}^{-1}\frac{\partial \mathbf{r}}{\partial \mathbf{y}} \tag{32}$$

- For solving a problem $\mathbf{r}\left(\mathbf{x}\right) = 0$, Newton iterates:

$$\mathbf{x} \leftarrow \mathbf{x} - \alpha\frac{\partial \mathbf{r}}{\partial \mathbf{x}}^{-1}\mathbf{r} \tag{33}$$

until $\mathbf{r}\left(\mathbf{x}\right) \approx 0$ where $\alpha \in [0, 1]$

- Runge-Kutta methods are described by:

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

$$\mathbf{K}_j = \mathbf{f}\left(\mathbf{x}_k + \Delta t\sum_{i=1}^{s} a_{ji}\mathbf{K}_i,\, \mathbf{u}\left(t_k + c_j\Delta t\right)\right), \quad j = 1, \dots, s \tag{34a}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t\sum_{i=1}^{s} b_i\mathbf{K}_i \tag{34b}$$

- For ERK methods, the relationship between the (minimum) number of stages s to the order o is given by:

s	1	2	3	4	6	7	9	11	...
o	1	2	3	4	5	6	7	8	...

Table 1: Stage to order of ERK methods

- Collocation methods use:

$$\dot{\mathbf{x}}(t_k + \Delta t \cdot \tau) \approx \dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau) = \sum_{i=1}^{s} \mathbf{K}_i \ell_i(\tau), \quad \tau \in [0,\, 1] \tag{35}$$

$$\mathbf{x}(t_k + \Delta t \cdot \tau) \approx \hat{\mathbf{x}}(t_k + \Delta t \cdot \tau) = \mathbf{x}_k + \Delta t \sum_{i=1}^{s} \mathbf{K}_i L_i(\tau) \tag{36}$$

where the Lagrange polynomials are given by:

$$\ell_i(\tau) = \prod_{j=1, j\neq i}^{s} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad \text{and} \quad L_i(\tau) = \int_0^{\tau} \ell_i(\xi) \mathrm{d}\xi \tag{37}$$

The Lagrange polynomials satisfy the conditions of

$$\text{Orthogonality:} \quad \int_0^1 \ell_i(\tau)\ell_j(\tau)\,\mathrm{d}\tau = 0 \quad \text{for} \quad i \neq j \tag{38a}$$

$$\text{Punctuality:} \quad \ell_i(\tau_j) = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases} \tag{38b}$$

and enforce the collocation equations (for $j = 1, \ldots, s$):

$$\dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau_j) = \mathbf{f}\left(\hat{\mathbf{x}}(t_k + \Delta t \cdot \tau_j),\, \mathbf{u}\left(t_k + \Delta t \cdot \tau_j\right)\right), \quad \text{in the explicit ODE case} \tag{39a}$$

$$\mathbf{F}\left(\dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau_j), \hat{\mathbf{x}}(t_k + \Delta t \cdot \tau_j),\, \mathbf{u}\left(t_k + \Delta t \cdot \tau_j\right)\right) = 0, \quad \text{in the implicit ODE case} \tag{39b}$$

$$\mathbf{F}\left(\dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau_j), \hat{\mathbf{z}}_j, \hat{\mathbf{x}}(t_k + \Delta t \cdot \tau_j),\, \mathbf{u}\left(t_k + \Delta t \cdot \tau_j\right)\right) = 0, \quad \text{in the fully-implicit DAE case} \tag{39c}$$

- Gauss-Legendre collocation methods select the set of points $\tau_{1,\ldots,s}$ as the zeros of the (shifted) Legrendre polynomial:

$$P_s\left(\tau\right) = \frac{1}{s!} \frac{\mathrm{d}^s}{\mathrm{d}\tau^s} \left[\left(\tau^2 - \tau\right)^s\right] \tag{40}$$

They achieve the order $\|\mathbf{x}_N - \mathbf{x}\left(t_{\mathrm{f}}\right)\| = \mathcal{O}\left(\Delta t^{2s}\right)$.

- Maximum-likelihood estimation is based on

$$\max_{\boldsymbol{\theta}} \quad \mathbb{P}\left[e_k = y_k - \hat{y}_k \quad \text{for} \quad k = 1, \ldots, N \mid \boldsymbol{\theta}\right] \tag{41}$$

If the noise sequence is uncorrelated, then

$$\mathbb{P}\left[e_k = y_k - \hat{y}_k \quad \text{for} \quad k = 0, \ldots, N \mid \boldsymbol{\theta}\right] = \prod_{k=1}^{N} \mathbb{P}\left[e_k = y_k - \hat{y}_k \mid \boldsymbol{\theta}\,\right] \tag{42}$$

- The solution of a linear least-squares problem

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{2} \|A\boldsymbol{\theta} - \mathbf{y}\|^2_{\Sigma_e^{-1}} \tag{43}$$

reads as:

$$\hat{\boldsymbol{\theta}} = \left(A^\top \Sigma_e^{-1} A\right)^{-1} A^\top \Sigma_e^{-1} \mathbf{y} \tag{44}$$

and the covariance of the parameter estimation based is given by the formula:

$$\Sigma_{\hat{\boldsymbol{\theta}}} = \left(A^\top \Sigma_e^{-1} A\right)^{-1} \tag{45}$$

- In system identification, given the a plant $G(z)$ and a noise $H(z)$ model description, the one-step-ahead predictor $\hat{y}(k|k-1)$ can be retrieved with

$$H(z)\hat{y}(z) = G(z)u(z) + (H(z) - 1)y(z) \tag{46}$$

- The Gauss-Newton approximation in an optimization problem

$$\min_{\mathbf{x}} \quad J(\mathbf{x}) = \frac{1}{2} \left\|\mathbf{R}(\mathbf{x})\right\|^2 \tag{47}$$

uses the approximation:

$$\frac{\partial^2 J}{\partial \mathbf{x}^2} \approx \frac{\partial R}{\partial \mathbf{x}}^\top \frac{\partial R}{\partial \mathbf{x}} \tag{48}$$

- The solution to an LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is given by:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\mathrm{d}\tau \tag{49}$$

and the transformation state-space to transfer function is given by:

$$G(s) = C\left(sI - A\right)^{-1} B + D \tag{50}$$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det(A) = ad - bc$
- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $det(A) = a.det(\begin{bmatrix} e & f \\ h & i \end{bmatrix}) - b.det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + c.det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det(A) = ad - bc$, $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- $\alpha = \mathbf{x}^T A\mathbf{x}$, where A is a symmetric matrix and $\mathbf{x}$ is $n \times 1$, A is $n \times n$, and A does not depend on $\mathbf{x}$, then, $\frac{\partial \alpha}{\partial \mathbf{x}} = 2\mathbf{x}^T A$.