Lösningar tenta 20/8 - 18, TMV138/181 (TMV130)

1(a) $$\frac{x(1-x)}{(x+2)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\Leftrightarrow x - x^2 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

$\underline{x=-1}$: $-1-1 = A(-1+2)^2 \Leftrightarrow A = -2$

$\underline{x=-2}$: $-2-4 = C(-2+1) \Leftrightarrow C = 6$

$$\Rightarrow x - x^2 + 2(x^2+4x+4) - 6x - 6 = B(x^2+3x+2)$$

$$\Rightarrow B = 1$$

$$\Rightarrow \int \frac{x(1-x)}{(x+2)^2(x+1)}\,dx = \int \left(\frac{1}{x+2} - \frac{2}{x+1} + \frac{6}{(x+2)^2}\right) dx =$$

$$= \ln|x+2| - 2\ln|x+1| - \frac{6}{x+1} + C$$

(b) $$\int \frac{e^{2x}}{e^{2x} - 2e^x + 2}\,dx = \left\{\begin{array}{l} t = e^x \\ dt = e^x\,dx \end{array}\right\} =$$

$$= \int \frac{t}{t^2 - 2t + 2}\,dt = \int \frac{t}{(t-1)^2+1}\,dt = \left\{\begin{array}{l} s = t-1 \\ ds = dt \end{array}\right\} =$$

$$= \int \frac{s+1}{s^2+1}\,ds = \frac{1}{2}\int \frac{2s}{s^2+1}\,ds + \int \frac{1}{s^2+1}\,ds =$$

$$= \frac{1}{2}\ln(s^2+1) + \arctan(s) = \frac{1}{2}\ln\big(\underbrace{(t-1)^2+1}_{t^2-2t+2}\big) + \arctan(t-1) =$$

$$= \frac{1}{2}\ln(e^{2x} - 2e^x + 2) + \arctan(e^x - 1) + C$$

2(a) $xy' = 1 - \frac{1}{\ln(x)} y \iff y' + \frac{1}{x\ln(x)} y = \frac{1}{x}$

$$\int \frac{1}{x\ln(x)} dx = \left\{ \begin{array}{l} t = \ln(x) \\ dt = \frac{1}{x} dx \end{array} \right\} = \int \frac{1}{t} dt = \ln|t| =$$

$$= \ln|\ln(x)| = \{x > 1\} = \ln(\ln(x))$$

$\Rightarrow$ Integr. faktor : $e^{\ln(\ln(x))} = \ln(x)$

$$\Rightarrow \frac{d}{dx}(\ln(x) y) = \frac{\ln(x)}{x} \Rightarrow$$

$$\Rightarrow \ln(x) \cdot y(x) = \int \frac{\ln(x)}{x} dx = \left\{ \begin{array}{l} t = \ln(x) \\ dt = \frac{1}{x} dx \end{array} \right\} =$$

$$= \int t\, dt = \frac{1}{2} t^2 = \frac{1}{2}(\ln(x))^2 + C$$

$$\therefore \quad y(x) = \frac{1}{2}\ln(x) + \frac{C}{\ln(x)}$$

(b) $y' = x\sqrt{4-y^2} \Rightarrow \int \frac{1}{\sqrt{4-y^2}} dy = \int x\, dx = \frac{x^2}{2} + C$

$$\int \frac{1}{\sqrt{4-y^2}} dy = \int \frac{1}{\sqrt{4} \cdot \sqrt{1 - \frac{y^2}{4}}} dy = \frac{1}{2} \int \frac{1}{\sqrt{1 - \left(\frac{y}{2}\right)^2}} dy =$$

$$= \frac{1}{2} \cdot 2 \arcsin\left(\frac{y}{2}\right) = \arcsin\left(\frac{y}{2}\right)$$

$$\Longrightarrow \arcsin\left(\frac{y}{2}\right) = \frac{x^2}{2} + C$$

$\underline{y(0) = -1}$: $\quad C = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$$\therefore \quad y(x) = 2\sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

3 (a) $\dfrac{\ln(1+x^2) - x\arctan(x)}{(\cos(x)-1)^2} =$

$$= \frac{\cancel{x^2} - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots - x\left(\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)}{\left(\cancel{1} - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots - \cancel{1}\right)^2} =$$

$$= \frac{\left(\frac{1}{3} - \frac{1}{2}\right)x^4 + \left(\frac{1}{3} - \frac{1}{5}\right)x^6 + \dots}{\frac{x^4}{4} - \frac{x^6}{4!} + \dots} =$$

$$= \frac{\cancel{x^4}\left(-\frac{1}{6} + \frac{2}{15}x^2 + \dots\right)}{\cancel{x^4}\left(\frac{1}{4} - \frac{1}{4!}x^2 + \dots\right)} \xrightarrow[x \to 0]{} \frac{-\frac{1}{6}}{\frac{1}{4}} = -\frac{2}{3}$$

(b) Om vi Maclaurinutvecklar $f$:s faktorer får vi

$$f(x) = \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right)\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots\right) =$$

$$= x^4 - \frac{x^6}{2} - \frac{x^6}{2} + \frac{x^8}{4} - \frac{x^8}{3!} + \frac{x^8}{3} + \dots =$$

$$= x^4 - x^6 + \left(\frac{1}{4} - \frac{1}{3!} + \frac{1}{3}\right)x^8 + \dots \qquad (*)$$

Å andra sidan vet vi att om vi ~~kan~~ potensserie-utveckla $f$ kring $x=0$, så är denna potensserie Maclaurinutvecklingen av $f$, dvs

$$(*) = f(0) + f'(0)x + \dots + \frac{f^{(6)}(0)}{6!}x^6 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$\Leftrightarrow \quad x^4 - x^6 + \left(\frac{7}{12} - \frac{1}{6}\right)x^8 + \dots =$$

$$= f(0) + f'(0)x + \dots + \frac{f^{(6)}(0)}{6!}x^6 + \dots$$

Koefficientidentifikation ger alltså att

$$\frac{f^{(6)}(0)}{6!} = -1 \Leftrightarrow f^{(6)}(0) = -6! = -720$$

4. (i) <u>Homogenlösning</u>: Kar. ekv.: $r^2+2r+2=0$

$$\Rightarrow r = -1 \pm \sqrt{1-2} = -1 \pm i$$

$$\Rightarrow y_h(x) = e^{-x}\left(A\cos(x) + B\sin(x)\right)$$

(ii) <u>Partikulärlösning</u>: Studera hjälpekv.

I. $v'' + 2v' + 2v = x + 1$

II. $u'' + 2u' + 2u = e^{ix}$

I. Ansätt $v(x) = Ax + B \Rightarrow v'(x) = A,\ v''(x) = 0$

$$\Rightarrow v'' + 2v' + 2v = 2A + 2Ax + 2B \overset{\text{vill}}{=} x + 1$$

$$\Rightarrow \begin{cases} 2A = 1 \Leftrightarrow A = 1/2 \\ 2A + 2B = 1 \Leftrightarrow B = \frac{1}{2} - A = 0 \end{cases}$$

$$\Rightarrow v_p(x) = \frac{1}{2}x$$

II. Låt $u = ze^{ix} \Rightarrow u' = z'e^{ix} + ize^{ix}$

$$u'' = z''e^{ix} + 2iz'e^{ix} - ze^{ix} \Rightarrow$$

$$\Rightarrow u'' + 2u' + 2u = (z'' + 2iz' - z + 2z' + 2iz + 2z)e^{ix} =$$

$$= \left(z'' + 2(1+i)z' + (1+2i)z\right)e^{ix} \overset{\text{vill}}{=} e^{ix}$$

$$\Rightarrow z'' + 2(1+i)z' + (1+2i)z = 1 \Rightarrow$$

$$\Rightarrow z_p = \frac{1}{1+2i} = \frac{1-2i}{5} = \frac{1}{5} - i\frac{2}{5}$$

$$\Rightarrow u_p = z_p e^{ix} = \left(\frac{1}{5} - i\frac{2}{5}\right)\left(\cos(x) + i\sin(x)\right) =$$

$$= \left(\frac{1}{5}\cos(x) + \frac{2}{5}\sin(x)\right) + i\left(\frac{1}{5}\sin(x) - \frac{2}{5}\cos(x)\right)$$

$$\Rightarrow y_p = v_p + \mathrm{Im}(u_p) = \frac{1}{2}x + \frac{1}{5}\sin(x) - \frac{2}{5}\cos(x)$$

$$\therefore y = y_h + y_p = Ae^{-x}\cos(x) + Be^{-x}\sin(x) + \frac{1}{2}x +$$

$$+ \frac{1}{5}\sin(x) - \frac{2}{5}\cos(x)$$

5. $$F'(x) = \frac{1-x}{(1+x^2)(x+1)}$$

$F'(x) = 0 \Rightarrow 1-x=0 \Leftrightarrow x=1$ min. eller max?

| | 0 | | 1 | | 2 |
|---|---|---|---|---|---|
| $F'$ | | $+$ | $0$ | $-$ | |
| $F$ | | $\nearrow$ | | $\searrow$ | |

$\Rightarrow x = 1$ max. pkt.

$$\Rightarrow F(1) = \int_0^1 \frac{1-t}{(1+t^2)(t+1)}\,dt$$

$$\frac{1-t}{(1+t^2)(t+1)} = \frac{At+B}{1+t^2} + \frac{C}{1+t} \Leftrightarrow$$

$$\Leftrightarrow 1-t = (At+B)(t+1) + C(1+t^2)$$

$\underline{t=-1:}$ $2 = 2C \Leftrightarrow C = 1$

$\underline{t=0:}$ $1 = B + C = B + 1 \Leftrightarrow B = 0$

$$\Rightarrow 1-t-(1+t^2) = At^2 + At \Rightarrow A = -1$$

$$\Rightarrow F(1) = \int_0^1 \left(\frac{-t}{t^2+1} + \frac{1}{t+1}\right) dt =$$

$$= -\frac{1}{2}\left[\ln(t^2+1)\right]_0^1 + \left[\ln|t+1|\right]_0^1 =$$

$$= -\frac{1}{2}\ln(2) + \ln(2) = \frac{1}{2}\ln(2)$$

6. Låt $y(x) = \sum_{n=0}^{\infty} a_n x^n$ . $|x|<R$ $\Rightarrow$

$\Rightarrow y'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} \Rightarrow xy' = \sum_{n=0}^{\infty} a_n n x^n$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} a_n (n-1) n x^{n-2} =$$

$$= \left\{ \begin{array}{l} k = n-2 \ , \ n=2 \Leftrightarrow k=0 \\ n = k+2 \ , \ n\to\infty \Leftrightarrow k\to\infty \end{array} \right\} = \sum_{k=0}^{\infty} a_{k+2}(k+1)(k+2)x^k$$

$\Rightarrow y'' + 2xy' + 2y =$

$$= \sum_{n=0}^{\infty} \left( a_{n+2}(n+1)(n+2) + 2a_n n + 2a_n \right) x^n \overset{\text{vill}}{=} 0$$

$\Longrightarrow a_{n+2}(n+1)(n+2) + 2a_n(n+1) = 0 \quad n = 0,1,2,\ldots$

$\Leftrightarrow a_{n+2} = -\dfrac{2a_n}{n+2} \quad n = 0,1,2,\ldots$

$y(0) = 1 \Leftrightarrow a_0 = 1$ , $y'(0) = 0 \Leftrightarrow a_1 = 0$

$a_2 = -\dfrac{2a_0}{2} = -\dfrac{2}{2}$ , $a_3 = -\dfrac{2a_1}{3} = 0$

$a_4 = -\dfrac{2a_2}{4} = (-1)^2 \dfrac{2^2}{2\cdot 4}$ , $a_5 = -\dfrac{2a_3}{5} = 0$

$a_6 = -\dfrac{2a_4}{6} = (-1)^3 \dfrac{2^3}{2\cdot 4\cdot 6}$ , $a_7 = 0$

$a_8 = -\dfrac{2a_6}{8} = (-1)^4 \cdot \dfrac{2^4}{2\cdot 4\cdot 6\cdot 8}$ , $a_9 = 0$

⋮

$$a_{2n} = (-1)^n \frac{2^n}{2^n \cdot n!} = \frac{(-1)^n}{n!} \quad , \quad a_{2n+1} = 0$$

$$\Rightarrow \quad y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

<u>Kvotkrit.:</u>

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right| =$$

$$= \lim_{n\to\infty} \frac{x^2}{n+1} = 0 \qquad \forall x \in \mathbb{R}$$

$$\therefore \quad y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \qquad \forall x \in \mathbb{R}$$

8. **Bevis**: Från beviset av integralkrit. vet vi att om $f$ kont., positiv & avtagande då $x \geq 1$, så

$$\int_1^n f(x)\,dx + f(n) \leq \sum_{k=1}^{n} f(k) \leq \int_1^n f(x)\,dx + f(1) \quad (*)$$

Låt $f(x) = \dfrac{1}{2x-1}$. Då $f$ kont. & positiv när $x \geq 1$

$$f'(x) = -\frac{2}{(2x-1)^2} \leq 0 \text{ så } f \text{ avtagande då } x \geq 1$$

$$\int_1^n f(x)\,dx = \int_1^n \frac{dx}{2x-1} = \frac{1}{2}\Big[\ln|2x-1|\Big]_1^n = \frac{1}{2}\ln(2n-1)$$

Detta ger att:

$$(*) \Leftrightarrow \frac{1}{2}\ln(2n-1) + \frac{1}{2n-1} \leq \sum_{k=1}^{n} f(k) \leq \frac{1}{2}\ln(2n-1) + 1$$

$$\Leftrightarrow \frac{\ln(2n-1)}{2\ln(n)} + \underbrace{\frac{1}{(2n-1)\ln(n)}}_{\xrightarrow[n\to\infty]{} 0} \leq \frac{1}{\ln(n)} \sum_{k=1}^{n} \frac{1}{2k-1} \leq$$

$$\leq \frac{\ln(2n-1)}{2\ln(n)} + \underbrace{\frac{1}{\ln(n)}}_{\xrightarrow[n\to\infty]{} 0}$$

$$\lim_{n\to\infty} \frac{\ln(2n-1)}{2\ln(n)} = \{\text{l'Hospital}\} = \lim_{n\to\infty} \frac{\frac{2}{2n-1}}{\frac{2}{n}} =$$

$$= \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$$

$$\therefore \lim_{n\to\infty} \frac{1}{\ln(n)} \sum_{k=1}^{n} \frac{1}{2k-1} = \frac{1}{2} \quad ∎$$