Lösningar tenta 4/4 – 18, TMV138/181

1(a) $\displaystyle\int \frac{\arctan(\sqrt{x})}{\sqrt{x}(1+x)}\,dx = \left\{\begin{array}{l} t=\sqrt{x} \\ x=t^2 \\ dx = 2t\,dt \end{array}\right\} =$

$$= \int \frac{\arctan(t)}{t(1+t^2)}\cdot 2t\,dt = 2\int \frac{\arctan(t)}{1+t^2}\,dt$$

$$\int \frac{\arctan(t)}{1+t^2}\,dt = \{p.i\} = (\arctan(t))^2 - \int \frac{\arctan(t)}{1+t^2}\,dt$$

$$\Leftrightarrow\quad 2\int \frac{\arctan(t)}{1+t^2}\,dt = (\arctan(t))^2$$

$$\therefore\quad \int \frac{\arctan(\sqrt{x})}{\sqrt{x}(1+x)}\,dx = (\arctan(\sqrt{x}))^2 + C$$

(b) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\,dx = \left\{\begin{array}{l} x = u^6 \\ dx = 6u^5\,du \end{array}\right\} =$

$$= \int \frac{1}{(u^6)^{1/2}(1+(u^6)^{1/3})}\,6u^5\,du = 6\int \frac{u^2}{1+u^2}\,du =$$

$$= 6\int\left(1 - \frac{1}{1+u^2}\right)du = 6(u - \arctan(u)) =$$

$$= \{u = x^{1/6}\} = 6x^{1/6} - 6\arctan(x^{1/6}) + C$$

2(a) $\left(1+\cos(x)\right)y' = (1+e^{-y})\sin(x) \Longrightarrow$

$$\Rightarrow \int \frac{1}{1+e^{-y}}\,dy = \int \frac{\sin(x)}{1+\cos(x)}\,dx$$

$$\int \frac{\sin(x)}{1+\cos(x)}\,dx = -\ln|\underbrace{1+\cos(x)}_{\geq 0}| = -\ln(1+\cos(x)) + C$$

$$\int \frac{1}{1+e^{-y}}\,dy = \int \frac{1}{1+e^{-y}}\cdot\frac{e^{y}}{e^{y}}\,dy = \int \frac{e^{y}}{e^{y}+1}\,dy = \ln(1+e^{y})$$

$$\Rightarrow \ln(1+e^{y}) = \ln\left(\frac{1}{1+\cos(x)}\right) + C \Rightarrow 1+e^{y} = \frac{C}{1+\cos(x)}$$

$\underline{y(0)=0}$: $\quad 1+1 = \dfrac{C}{1+1} \iff C = 4$

$$\therefore\ y(x) = \ln\left(\frac{4}{1+\cos(x)} - 1\right)$$

(b) Derivera båda leden m.a.p. x & använd :

$$f'(x) + 2x = 4x f(x) \iff f'(x) - 4x f(x) = -2x$$

$$\int -4x\,dx = -2x^{2} \Rightarrow \text{Integr. faktor: } e^{-2x^{2}}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-2x^{2}} f(x)\right) = -2x e^{-2x^{2}} \Rightarrow$$

$$\Rightarrow e^{-2x^{2}} f(x) = \int -2x e^{-2x^{2}}\,dx = \frac{1}{2}\int -4x e^{-2x^{2}}\,dx =$$

$$= \frac{1}{2} e^{-2x^{2}} + C$$

$$\Rightarrow f(x) = \frac{1}{2} + C e^{2x^{2}}$$

Låt $x = 0$ i integralekv.:

$$f(0) + 0^2 - 1 = \int_0^0 4t f(t)\,dt = 0 \iff f(0) = 1$$

$$\Rightarrow \quad 1 = \frac{1}{2} + C \iff C = \frac{1}{2}$$

$$\therefore \; f(x) = \frac{1 + e^{2x^2}}{2}$$

3(a) $$\frac{\cos^2(x)-e^{-x^2}}{\ln(1+x^4)}=\frac{\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots\right)^2-\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{3!}+\dots\right)}{x^4-\frac{x^8}{2}+\frac{x^{12}}{3}-\dots}=$$

$$=\frac{\cancel{1}-\cancel{x^2}+\frac{x^4}{4}+2\cdot\frac{x^4}{4!}+\dots-\cancel{1}+\cancel{x^2}-\frac{x^4}{2}+\frac{x^6}{6}-\dots}{x^4-\frac{x^8}{2}+\frac{x^{12}}{3}-\dots}=$$

$$=\frac{\left(\frac{1}{4}+\frac{2}{24}-\frac{1}{2}\right)x^4+\frac{x^6}{6}+\dots}{x^4\left(1-\frac{x^4}{2}+\frac{x^8}{3}-\dots\right)}=$$

$$=\frac{\cancel{x^4}\left(\frac{3+1-6}{12}+\frac{1}{6}x^2+\dots\right)}{\cancel{x^4}\left(1-\frac{x^4}{2}+\frac{x^8}{3}-\dots\right)}\xrightarrow[x\to 0]{}-\frac{1}{6}$$

(b) $$\frac{\ln\left(\sqrt{1+x^2}\right)-(1-\cos(x))}{(e^{x^2}-1)\sin^2(x)}=\frac{\frac{1}{2}\ln(1+x^2)+\cos(x)-1}{(e^{x^2}-1)\sin^2(x)}=$$

$$=\frac{\frac{1}{2}\left(\cancel{x^2}-\frac{x^4}{2}+\frac{x^6}{3}-\dots\right)+\cancel{1}-\cancel{\frac{x^2}{2}}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots-\cancel{1}}{\left(\cancel{1}+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\dots-\cancel{1}\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)^2}=$$

$$=\frac{\left(\frac{1}{4!}-\frac{1}{4}\right)x^4+\left(\frac{1}{6}-\frac{1}{6!}\right)x^6+\dots}{\left(x^2+\frac{x^4}{2}+\frac{x^6}{6}+\dots\right)\left(x^2-\frac{2}{3!}x^4+\dots\right)}=$$

$$=\frac{\left(\frac{1}{24}-\frac{1}{4}\right)x^4+\left(\frac{1}{6}-\frac{1}{6!}\right)x^6+\dots}{x^4+\left(\frac{1}{2}-\frac{2}{6}\right)x^6+\dots}=$$

$$= \frac{\cancel{x^4}\left(-\frac{5}{24} + \left(\frac{1}{6} - \frac{1}{6!}\right)x^2 + \dots\right)}{\cancel{x^4}\left(1 + \left(\frac{1}{2} - \frac{1}{3}\right)x^2 + \dots\right)} \xrightarrow[x \to 0]{} -\frac{5}{24}$$

4 (i) Homogenlösning: Kar.ekv.: $r^2+9=0 \Rightarrow r=\pm 3i$

$$\Rightarrow y_h(x) = A\cos(3x) + B\sin(3x)$$

(ii) Partikulärlösning: Studera hjälpekv.

$$u'' + 9u = 9e^{3ix}$$

Låt $u = ze^{3ix} \Rightarrow u' = z'e^{3ix} + 3ize^{3ix}$

$$u'' = z''e^{3ix} + 6iz'e^{3ix} - 9ze^{3ix}$$

$$\Rightarrow u'' + 9u = (z'' + 6iz' - 9z + 9z)e^{3ix} =$$

$$= (z'' + 6iz')e^{3ix} \stackrel{\text{vill}}{=} 9e^{3ix}$$

$$\Rightarrow z'' + 6iz' = 9 \Rightarrow z_p = \frac{9x}{6i} = -\frac{3i}{2}x \Rightarrow$$

$$\Rightarrow u_p = z_p e^{3ix} = -\frac{3i}{2}x(\cos(3x) + i\sin(3x)) =$$

$$= \frac{3}{2}x\sin(3x) - i\frac{3}{2}x\cos(3x)$$

$$\Rightarrow y_p(x) = \mathrm{Re}(u_p) = \frac{3}{2}x\sin(3x)$$

$$\Rightarrow y = y_h + y_p = A\cos(3x) + B\sin(3x) + \frac{3}{2}x\sin(3x)$$

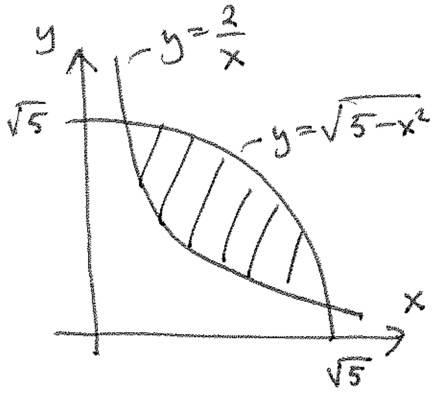
(iii) Begynnelsevillkor: $y(0) = 0 \Leftrightarrow A = 0$

$$y'(x) = 3B\cos(3x) + \frac{3}{2}\sin(3x) + \frac{9}{2}x\cos(3x)$$

$$y'(0) = 1 \Leftrightarrow 3B = 1 \Leftrightarrow B = \frac{1}{3}$$

$$\therefore y(x) = \left(\frac{1}{3} + \frac{3}{2}x\right)\sin(3x)$$

5 (a) $\sqrt{5-x^2} = \frac{2}{x} \Rightarrow$



$$\Rightarrow 5-x^2 = \frac{4}{x^2} \Leftrightarrow 5x^2 - x^4 = 4$$

$$\Leftrightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow \{t = x^2\} \Rightarrow$$

$$\Rightarrow t^2 - 5t + 4 = 0$$

$$\Rightarrow t = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 4} = \frac{5}{2} \pm \frac{3}{2} \Rightarrow t_1 = 1,\ t_2 = 4$$

$$\Rightarrow x_{1,2} = \overset{+}{(-)}1 \ ,\ x_{3,4} = \overset{+}{(-)}2$$

$$\Rightarrow V_x = \int_1^2 \pi\left((\sqrt{5-x^2})^2 - \left(\frac{2}{x}\right)^2 \right) dx =$$

$$= \pi \int_1^2 \left(5 - x^2 - \frac{4}{x^2}\right) dx = \pi \left[5x - \frac{x^3}{3} + \frac{4}{x}\right]_1^2 =$$

$$= \pi\left(10 - \frac{8}{3} + 2 - \left(5 - \frac{1}{3} + 4\right)\right) = \frac{2\pi}{3} \text{ v.e.}$$

(b) $y' = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow (y')^2 = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$

$$\Rightarrow 1 + (y')^2 = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\Rightarrow S_y = \int_1^4 2\pi x \sqrt{1 + (y')^2}\, dx =$$

$$= 2\pi \int_1^4 x \left| \frac{x^2}{4} + \frac{1}{x^2} \right| dx = 2\pi \int_1^4 \left(\frac{x^3}{4} + \frac{1}{x}\right) dx =$$

$$= 2\pi \left[\frac{x^4}{16} + \ln|x|\right]_1^4 = \frac{255\pi}{8} + 4\pi \ln(2) \text{ a.e.}$$

6. $$\frac{a(\ln(1+x))^3}{(b-\cos(x))\sin(x)} = \frac{a\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right)^3}{\left(b-1+\frac{x^2}{2}-\frac{x^4}{4!}+\ldots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)} =$$

$$= \frac{a\left(x^3-\frac{3}{2}x^4+\ldots\right)}{(b-1)x-\frac{(b-1)}{6}x^3+\frac{1}{2}x^3-\frac{x^5}{12}+\ldots}$$

Ser att för att ett gränsvärde ska kunna existera då $x \to 0$, måste $b=1$

$$\Rightarrow \frac{a\left(x^3-\frac{3}{2}x^4+\ldots\right)}{\frac{1}{2}x^3-\frac{x^5}{12}+\ldots} = \frac{a\cancel{x^3}\left(1-\frac{3}{2}x+\ldots\right)}{\cancel{x^3}\left(\frac{1}{2}-\frac{x^2}{12}+\ldots\right)} =$$

$$= \frac{a\left(1-\frac{3}{2}x+\ldots\right)}{\frac{1}{2}-\frac{x^2}{12}+\ldots} \xrightarrow[x\to 0]{} 2a = 1 \quad \text{om } a=\frac{1}{2}$$

$\therefore\ a=\frac{1}{2}$ och $b=1$

8. Bevis: Vet att

(i) e^x växande funktion. dvs $x_1 < x_2 \Rightarrow e^{x_1} \leq e^{x_2}$

(ii) $f(x) \leq g(x) \quad \forall x \in [a, b] \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

$$-1 \leqslant \sin(x) \leqslant 1 \qquad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\overset{(i)}{\Rightarrow} \quad e^{-1} \leqslant e^{\sin(x)} \leqslant e^{1} \qquad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\overset{(ii)}{\Rightarrow} \quad \underbrace{\int_{-\pi/2}^{\pi/2} e^{-1}\,dx}_{=\frac{1}{e}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} \leqslant \int_{-\pi/2}^{\pi/2} e^{\sin(x)}\,dx \leqslant \underbrace{\int_{-\pi/2}^{\pi/2} e\,dx}_{=e\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}$$

$$\therefore \quad \frac{\pi}{e} \leqslant \int_{-\pi/2}^{\pi/2} e^{\sin(x)}\,dx \leqslant \pi e \qquad \blacksquare$$