p1

$$Z = \frac{Z_1}{Z_2} \qquad Z_1 = (3+2i^{19})(2-3i^{1015})$$

$$Z_2 = (1+i)(1-2i)$$

$$i^{19} = i^{4\cdot4+3} = (i^4)^4 \cdot i^3 = 1^4 \cdot i^3 = i^2 \cdot i = -i$$

$$1015:4 = (1000+15)/4 = 250 + \frac{15}{4} = 250 + \frac{12+3}{4}$$

$$= 250 + \frac{12}{4} + \frac{3}{4} = 250 + 3 + \frac{3}{4} = 253 + \frac{3}{4}$$

$$1015 = 253 \cdot 4 + 3 \quad \Rightarrow \quad i^{1015} = (i^4)^{253} \cdot i^3 = i^3 = -i$$

$$Z_1 = [3+2(-i)][2-3(-i)] = (3-2i)(2+3i) =$$

$$= 3\cdot2 + 3\cdot3\cdot i - 2i2 - 6i^2 = 6+9i-4i-6(-1)$$

$$= 6+6+5i = 12+5i$$

$$Z_2 = (1+i)(1-2i) = 1-2i+i-2i^2 = 1+2-2i+i$$

$$= 3-i$$

$$\frac{Z_1}{Z_2} = \frac{12+5i}{3-i} \cdot 1 = \frac{(12+5i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{36+12i+15i+5i^2}{9+1} = \frac{36-5+27i}{10}$$

$$\boxed{Z = \frac{31+27i}{10}} \qquad \operatorname{Re} Z = \frac{31}{10} \quad \operatorname{Im} Z = \frac{27}{10}$$

p2

$\Delta(x^2 \ln x - x\cos\pi x) = ?$

$y = x^2 \ln x - x\cos\pi x$

$x_1 = 1 \quad y_1 = 1^2 \ln 1 - 1\cos(\pi\cdot 1) = 1\cdot 0 - 1\cos\pi = 0-(-1)$
$= 1$

$x_2 = 1.1 \quad y_2 = ? \qquad y_2 = y_1 + \Delta y \qquad \Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

$\Delta y = (y')_{x=1} \cdot \Delta x \qquad \Delta x = 1.1 - 1 = 0.1$

$y' = 2x\ln x + x^2\frac{1}{x} - \cos\pi x - x(-\sin\pi x \cdot \pi)$
$= 2x\ln x + x - \cos\pi x + \pi x \sin\pi x$

$(y')_{x=1} = 2\cdot 1\cdot \underbrace{\ln 1}_{0} + 1 - \cos\pi + \pi \underbrace{\sin\pi}_{0}$
$= 1 - (-1) = 2$

$\Delta y = 2\cdot 0.1 = 0.2$ y ökar!

p3/1

$$f(x) = \begin{cases} \dfrac{x^{101} + x^{50} - 2}{x^3 - 1} & x \neq 1 \\ c & x = 1 \end{cases}$$

För $x \neq 1$ $f(x)$ är en rationell funktion. f är kontinuerlig (sats i ADAMS) för varje x sådan att täljaren är inte noll. $x^3 - 1$ är bara noll i $x = 1$!

varför? ju för att $x^3 - 1 = (x-1)(x^2 + x + 1)$

och polynomet $x^2 + x + 1$ har inga reela noll punkter:

$$x^2 + x + 1 = 0$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 1}}{2 \cdot 1} = \frac{-1 \pm \sqrt{-3}}{2}$$

detta betyder att $f(x)$ är kontinuerlig i varje $x \neq 1$

hur löser vi $x = 1$ fallet? om $f(x)$ skall vara kontinuerlig i $x = 1$ det måste vara sant att

$$\lim_{x \to 1} f(x) = f(1)$$

vi måste räkna HS och VS separat och justera att vi kan (alltså konstanten c)

Så att VS = HS

$$HS. \quad f(1) = \begin{cases} \cancel{\vdots \quad x \neq 1} \\ c \quad x = 1 \end{cases} = c$$

$$VS. \quad \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^{101} + x^{50} - 2}{x^3 - 1} = \left\{ \frac{0}{0}, LH \right\}$$

$$= \lim_{x \to 1} \frac{101\, x^{100} + 50 x^{49} - 0}{3x^2 - 0}$$

$$= \frac{101 + 50}{3} = \frac{151}{3}$$

$$\boxed{\text{SVAR Ja, funktionen blir kontinuerlig i } \mathbb{R} \text{ om vi väljer } c = \frac{151}{3}.}$$

p4

$$\lim_{h\to 0} \frac{h\,e^{(7+h)^2} - h\,e^{49}}{h^2} = \lim_{h\to 0} \frac{h\left[e^{(7+h)^2} - e^{7^2}\right]}{h^2}$$

$$= \lim_{h\to 0} \frac{e^{(7+h)^2} - e^{7^2}}{h} = f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

jämför och välj så att de matchar

$f(x) = e^{x^2}$ $\quad x_0 = 7$

$$f'(x_0) = \left(e^{x^2} \cdot 2x\right)\Big|_{x=x_0=7} = 2 \cdot 7 \cdot e^{7^2} = \underbrace{14 \cdot e^{49}}_{\text{SVAR!}}$$

p5

$$f(x) = e^x \ln x - x^2 \sin x + 7x^3 + 8 \quad , \quad f''(x) = ?$$

$$f'(x) = (e^x)' \ln x + e^x (\ln x)' - (x^2)' \sin x - x^2 (\sin x)'$$
$$+ 7\,(x^3)' + 8'$$

$$= e^x \ln x + e^x \frac{1}{x} - 2x \sin x - x^2 \cos x + 7 \cdot 3x^2 + 0$$

$$\boxed{f'(x) = e^x \ln x + \frac{e^x}{x} - 2x \sin x - x^2 \cos x + 21x^2}$$

$$f''(x) = e^x \ln x + e^x \frac{1}{x} + \frac{e^x \cdot x - e^x \cdot 1}{x^2}$$
$$-2(\sin x + x \cos x) - (2x \cos x + x^2(-\sin x))$$
$$+ 21 \cdot 2 \cdot x$$

$$= e^x \ln x + 2\frac{e^x}{x} - \frac{e^x}{x^2}$$
$$- 2 \sin x - 2x \cos x - 2x \cos x + x^2 \sin x$$
$$+ 42x$$

$$\boxed{f''(x) = e^x \ln x + e^x\left(\frac{2}{x} - \frac{1}{x^2}\right) - 2 \sin x - 4x \cos x + x^2 \sin x + 42x}$$

p6

$\frac{dy}{dx} = ?$ $x = 1$ $y = 1$

$2x = \frac{1}{y} + y^4$ $\Big|$ $\frac{d}{dx}(\ ) = (\ )'$

$2 = -\frac{1}{y^2} y' + 4y^3 y'$

använd $x = 1,\ y = 1$

$2 = -\frac{1}{1^2} y' + 4 \, 1^3 \, y'$

$2 = -y' + 4y' = +3y'$

$$\boxed{y' = \frac{2}{3}}$$

p7

$$I_2 = \int_0^{\pi} [u^3 - \sin u]\, du \qquad I_3 = \int_0^{\pi} \cos\sqrt{u}\, du$$

$$I_1 = \int_0^1 \left[(\pi x)^3 - \sin(\pi x) + 15 \cos\left(\sqrt{\pi x}\right) \right] dx = \left\{ \begin{array}{l} \pi x = u \\ \pi\, dx = du \end{array} \right\}$$

$$= \int_0^{\pi} [\underbrace{u^3 - \sin u}_{I_2} + \underbrace{15 \cos\sqrt{u}}_{I_3}] \frac{du}{\pi}$$

$$= \frac{1}{\pi} \int_0^{\pi} (u^3 - \sin u)\, du + \frac{15}{\pi} \int_0^{\pi} \cos\sqrt{u}\, du$$

$$= \frac{1}{\pi} I_2 + \frac{15}{\pi} I_3$$

$$\boxed{a = \frac{1}{\pi} \qquad b = \frac{15}{\pi}}$$

p8

$$I = \int_0^{\pi} x^2 \sin x \, dx$$

$$\int_a^b u \cdot dv = uv\Big|_a^b - \int_a^b du \cdot v = u \cdot v\Big|_a^b - \int_a^b v \, du$$

detta är ett bra sätt att komma ihåg

två val

a) $u = x^2 \quad \Rightarrow \quad du = 2x\,dx$
$dv = \sin x \, dx \qquad v = -\cos x$

KAN ANVÄNDAS

b) $u = \sin x \quad \Rightarrow \quad du = \cos x \, dx$
$dv = x^2 dx \qquad v = \frac{x^3}{3}$

INTE BRA

$$I = \int_0^{\pi} x^2 \sin x \, dx = \left\{ \begin{array}{l} u = x^2, \; du = 2x\,dx \\ dv = \sin x \, dx, \; v = -\cos x \end{array} \right\} =$$

$$= x^2(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} (-\cos x) 2x \, dx$$

$$= x^2 \cos x \Big|_{\pi}^{0} + 2 \int_0^{\pi} x \cos x \, dx$$

$$\int_0^{\pi} x \cos x \, dx = \left\{ \begin{array}{l} u = x, \; du = dx \\ dv = \cos x \, dx, \; v = \sin x \end{array} \right\} =$$

$$= x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x \, dx$$

$$= (x \sin x + \cos x)\Big|_0^{\pi}$$

$$I = \left[-x^2 \cos x + 2(x \sin x + \cos x) \right]\Big|_0^{\pi}$$

$$I = \Big[- x \cos x + 2 (x \sin x \ldots) \Big]_0$$

$$= \Big[(2 - x^2) \cos x + 2x \sin x \Big]_0^{\pi}$$

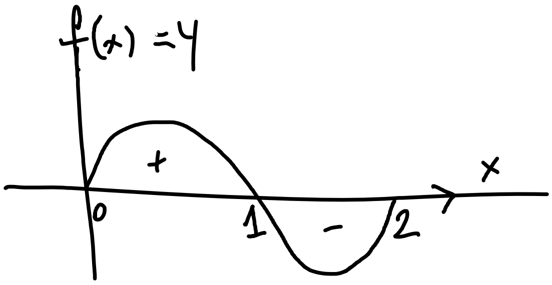
$$= (2 - \pi^2) \underbrace{\cos \pi}_{-1} + 2\pi \cancel{\sin \pi}^{0}$$

$$- \Big\{ (2 - 0^2) \underbrace{\cos 0}_{1} + 2 \cdot 0 \cancel{\sin 0}^{0} \Big\}$$

$$= (2 - \pi^2)(-1) - 2 = -2 + \pi^2 - 2 = \pi^2 - 4$$

$$\boxed{I = \pi^2 - 4}$$

p9



Det är inte sant att kartongens area är $\int_0^2 f(x)\,dx$ trots att detta integral definierar ytan under grafen

Man måste komma ihåg här att ytan under grafen kan dels vara positiv och dels negativ: "+" och "−" bitar kan annulera varandra

Vi är intreserade i kartongens area.

$$\text{ytan under grafen} = A_G = \int_0^2 f(x)\,dx$$ kan vara 0 i princip

Den rätta formlen är:

$$\text{kartongens area} = A_K = \int_0^2 |f(x)|\,dx$$ borde inte vara noll

f(x) byter tecken i $x=1$

$$\int_0^2 |f(x)|\,dx = \underbrace{\int_0^1 f(x)\,dx}_{A_1} + \underbrace{\int_1^2 (-f(x))\,dx}_{A_2}$$

$$A_1 = \int_0^1 x(x-1)(x-2)\,dx = \int_0^1 x(x^2-2x-x+2)\,dx$$

$$= \int_0^1 x(x^2-3x+2)\,dx = \int_0^1 (x^3-3x^2+2x)\,dx$$

$$= \int_0^1 x(x^2-3x+2)\,dx = \int_0^1 (x^3-3x^2+2x)\,dx$$

$$= \left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_0^1 = \frac{1}{4} - 1 + 1 = \frac{1}{4}$$

$$A_2 = \int_1^2 (-f(x))\,dx = -\int_1^2 f(x)\,dx = \dots =$$

$$= -\left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_1^2 = \left(\frac{x^4}{4} - x^3 + x^2\right)\Big|_2^1$$

$$= \frac{1}{4} - 1 + 1 - \left(\frac{2^4}{4} - 2^3 + 2^2\right)$$

$$= \frac{1}{4} - \left(\frac{4 \cdot 4}{4} - 8 + 4\right) = \frac{1}{4} - (4 - 8 + 4)$$

$$= \frac{1}{4}$$

$$\boxed{A_K = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}}$$

en viktig kommentar: vi ser att ytan under grafen är 0, men kartongens yta är förstås större än noll.

p10

$$\lim_{x\to 0} \frac{\cos(2x) - \cos(3x)}{x^2} = \left\{ \frac{1-1}{0} = \frac{0}{0}, \text{ LH} \right\}$$

$$= \lim_{x\to 0} \frac{-\sin(2x)2 + \sin(3x)3}{2x} = \left\{ \frac{0+0}{0} = \frac{0}{0}, \text{ LH} \right\}$$

$$= \lim_{x\to 0} \frac{-2\cos(2x)\cdot 2 + 3\cos(3x)\cdot 3}{2}$$

$$= \frac{-2\cdot 1\cdot 2 + 3\cdot 1\cdot 3}{2} = \frac{9-4}{2} = \frac{5}{2}$$

SVAR: $\frac{5}{2}$

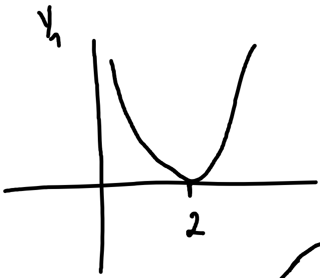
p11

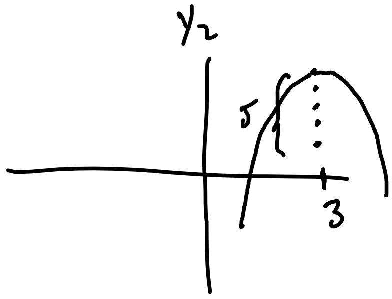
$y_1 = (x-2)^2$

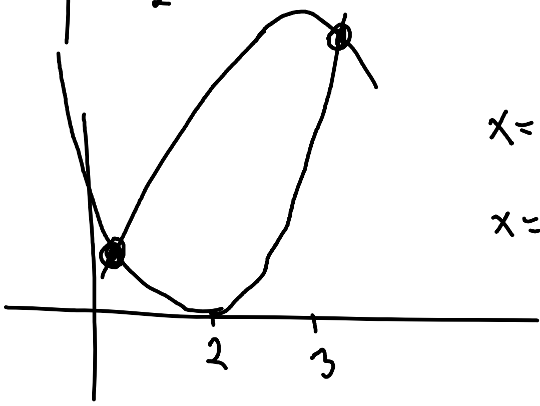
$y_2 = -4 + 6x - x^2 = -(x^2 - 6x + 4)$

$= -[(x-3)^2 - 9 + 4] = -(x-3)^2 + 9 - 4$

$= -(x-3)^2 + 5$







$x = 3 \quad y_1 = (3-2)^2 = 1$

$x = 3 \quad y_2 = -(3-3)^2 + 5 = 5$

$y_2 > y_1$ när $x = 3$

a) hittar skärningspunkter först

för vilken x är $y_1 = y_2$?

$(x-2)^2 = -4 + 6x - x^2$

$x^2 - 4x + 4 = -4 + 6x - x^2$

$2x^2 - 10x + 8 = 0 \quad | :2$

$x^2 - 5x + 4 = 0$

$ax^2 + bx + c = 0$

$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x_{1,2} = \frac{-(-5) \pm \sqrt{25 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$

$= \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$

$= \begin{cases} \frac{5+3}{2} = \frac{8}{2} = 4 \\ \frac{5-3}{2} \end{cases}$

$$= \frac{2}{2} = 1$$

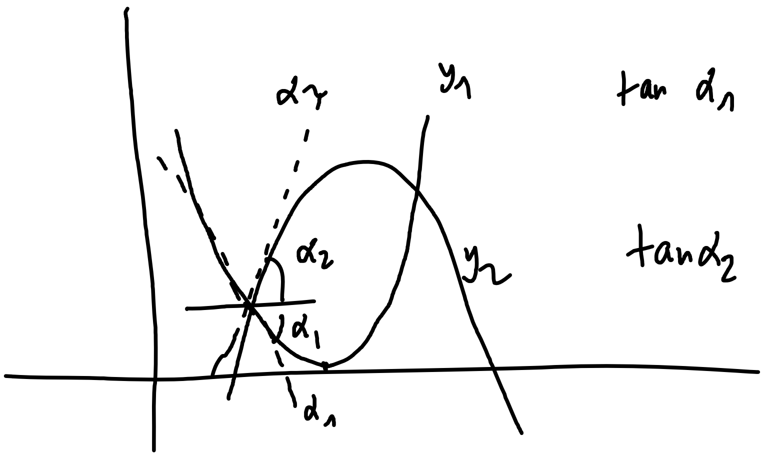
Kollar $x = 1$ $\quad y_1 = (1-2)^2 = 1$ ✓

$y_2 = -4 + 6 - 1 = 6 - 5 = 1$ ✓

$x = 4$ $\quad y_1 = (4-2)^2 = 2^2 = 4$ ✓

$y_2 = -4 + 6 \cdot 4 - 4^2 = -4 + 24 - 16 = 24 - 20$

$= 4$ ✓



$$\tan \alpha_1 = \left.\frac{dy_1}{dx}\right|_{x=1}$$

$$\tan \alpha_2 = \left.\frac{dy_2}{dx}\right|_{x=1}$$

$$\tan \alpha_1 = 2(x-2)\Big|_{x=1} = 2(1-2) = -2 \equiv k_1$$

$$\tan \alpha_2 = (6-2x)\Big|_{x=1} = 6 - 2 \cdot 1 = 4 \equiv k_2$$

$$\alpha_1 = \arctan k_1 \quad , \quad \alpha_2 = \arctan k_2$$

$$\tan(\alpha_2 - \alpha_1) = ?$$

$$\tan(\alpha_2 - \alpha_1) = \frac{\sin(\alpha_2 - \alpha_1)}{\cos(\alpha_2 - \alpha_1)} = \frac{\sin\alpha_2 \cos\alpha_1 - \cos\alpha_2 \sin\alpha_1}{\cos\alpha_2 \cos\alpha_1 + \sin\alpha_2 \cdot \sin\alpha_1}$$

$$= \frac{\frac{\sin\alpha_2}{\cos\alpha_2} - \frac{\sin\alpha_1}{\cos\alpha_1}}{1 + \frac{\sin\alpha_2}{\cos\alpha_2}\frac{\sin\alpha_1}{\cos\alpha_1}} = \frac{k_2 - k_1}{1 + k_2 k_1}$$

$$= \frac{4 - (-2)}{\dots} = \frac{4+2}{\dots} = -\frac{6}{7}$$

$$= \frac{4-(-2)}{1+4(-2)} = \frac{4+2}{1-8} = -\frac{6}{7}$$

$$\boxed{x=1 \qquad \tan(\alpha_2-\alpha_1) = -\frac{6}{7}}$$

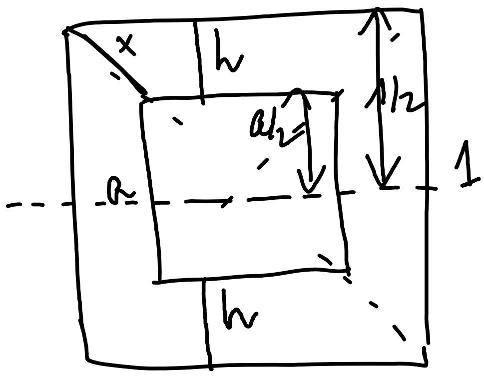
uträkning i $x=4$ sker på samma sätt och ger samma svar

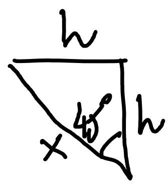
p12

$2h + a = 1$

$\Uparrow$

$h + a + h = 1$





$h^2 + h^2 = x^2$

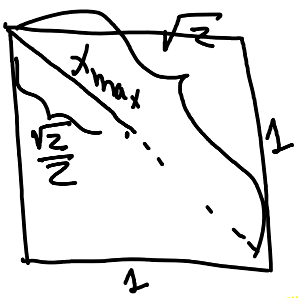
$2h^2 = x^2$

$h = \frac{x}{\sqrt{2}}$

$2\frac{x}{\sqrt{2}} + a = 1$

$\Downarrow$

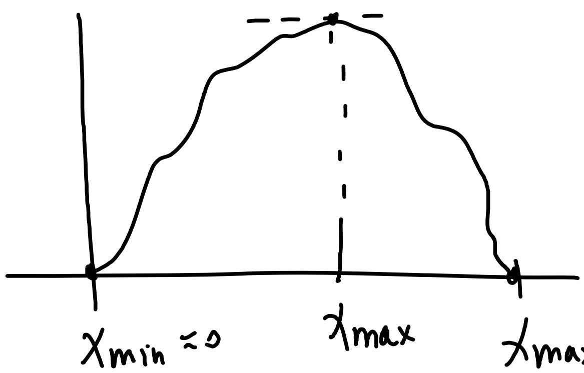
$a = 1 - \frac{2}{\sqrt{2}}x \Rightarrow a = 1 - \sqrt{2}x$



$x_{min} = 0$ $\qquad a_{max} = 1 - \frac{2}{\sqrt{2}} \cdot 0 = 1$ ✓

$x_{max} = \frac{\sqrt{2}}{2}$ $\qquad a_{min} = 1 - \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{2} = 1 - 1 = 0$ ✓

$V(x) = h(x) \cdot a(x)^2 = \frac{x}{\sqrt{2}}\left(1 - \sqrt{2}x\right)^2$



$x_{max} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$V'(x_{max}) = 0$; $\qquad 0 = \frac{d}{dx}\left(\frac{x}{\sqrt{2}}\left(1 - \sqrt{2}x\right)^2\right)$

$$0 = \frac{d}{dx}\left(x(1-\sqrt{2}x)^2\right) = 1(1-\sqrt{2}x)^2 + x2(1-\sqrt{2}x)(-\sqrt{2})$$

$$0 = 1 - 2\sqrt{2}x + 2x^2 - 2\sqrt{2}x + 4x^2$$

$$0 = 1 - 4\sqrt{2}x + 6x^2$$

$$x_{1,2} = \frac{-(-4\sqrt{2}) \pm \sqrt{16\cdot 2 - 4\,1\cdot 6}}{2\cdot 6}$$

$$= \frac{4\sqrt{2} \pm \sqrt{32-24}}{2\cdot 6} = \frac{4\sqrt{2} \pm \sqrt{8}}{2\cdot 6} =$$

$$= \frac{4\sqrt{2} \pm \sqrt{4\cdot 2}}{2\cdot 6} = \frac{4\sqrt{2} \pm 2\sqrt{2}}{2\cdot 3\cdot 2} = \frac{4\pm 2}{2\cdot 3\cdot 2}\sqrt{2}$$

$$= \frac{2\pm 1}{6}\sqrt{2} = \begin{cases} \frac{3}{6}\sqrt{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \longrightarrow x_{max} \\ \frac{1}{6}\sqrt{2} = \frac{\sqrt{2}}{6} \end{cases}$$

$$\boxed{x_{max} = \frac{\sqrt{2}}{6}}$$

$$V_{max} = \frac{x}{\sqrt{2}}(1-\sqrt{2}x)^2\Big|_{x=x_{max}} = \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\,6}\left(1-\sqrt{2}\,\frac{\sqrt{2}}{6}\right)^2$$

$$= \frac{1}{6}\left(1-\frac{2}{6}\right)^2 = \frac{1}{6}\left(1-\frac{1}{3}\right)^2 = \frac{1}{6}\,\frac{2^2}{3^2} =$$

$$= \frac{\cancel{2}\cdot 2}{\cancel{2}\cdot 3\cdot 3\cdot 3} = \boxed{\frac{2}{27}}$$

p13

$$x^2 + x = xy + x^3 y^4 \quad (*)$$

$$x_1 = 1, \; y_1 = 1 \qquad 1^2 + 1 = 1 \cdot 1 + 1^3 \cdot 1^4$$
$$2 = 2 \quad \checkmark$$

ekvationen (*) definierar y som funktion av x i implicit form: $y = f(x)$. Vi vet att $y_1 = f(x_1)$. Nu ändras x från x_1 till $x_2 = x_1 + \Delta x$. Vi kan använda Taylor utveckling

$$y_2 = f(x_2) = f(x_1 + \Delta x) = f(x_1) + f'(x_1) \cdot \Delta x + \mathcal{O}(\Delta x^2)$$

$$\boxed{y_2 \approx y_1 + f'(x_1)\, \Delta x}$$

Hur kan vi räkna $f'(x_1)$? Vi vet att

$$f'(x_1) \equiv \left.\frac{dy}{dx}\right|_{\substack{x = x_1 \\ y = y_1}}$$

Med implicit derivata tekniken får man att

$$\frac{d}{dx}\left(x^2 + x\right) = \frac{d}{dx}\left(xy + x^3 y^4\right)$$

$$2x + 1 = 1y + x\frac{dy}{dx} + 3x^2 y^4 + x^3 4y^3 \frac{dy}{dx}$$

använd $x = x_1 = 1$ och $y = y_1 = 1$

$$2\cdot 1 + 1 = 1\cdot 1 \quad 1\frac{dy}{dx} + 3\cdot 1^2 1^4 + 1^3 4\, 1^3 \frac{dy}{dx}$$

$$3 = 1 + \frac{dy}{dx} + 3 + 4\frac{dy}{dx}$$

$$3 - 1 - 3 = (1+4)\frac{dy}{dx}$$

$$-1 = 5\frac{dy}{dx}, \qquad \frac{dy}{dx} = -\frac{1}{5}$$

$$f'(x_1) = -\frac{1}{5}$$

$$y_2 = y_1 + f'(x_1)\cdot \Delta x$$

$$y_2 = 1 - \frac{1}{5}\, 0{,}1 = 1 - \frac{1}{5\cdot 10} = 1 - \frac{1}{50}$$

$$\boxed{y_2 \approx \frac{49}{50}}$$

och man kan fortsätta till högre grad av TU. Tex det går att hitta kvadratisk approximation till y2 med följande utvecklingen:

$$y_2 \approx f(x_1) + f'(x_1)\,\Delta x + \frac{1}{2} f''(x_1)\,\Delta x^2$$

hur kan man räkna $f''(x_1)$? Ja, med att derivera två gånger ekvationen som definierar $f(x)$ implicit:

$$x^2 + x = xy + x^3y^4 \quad \Big| \frac{d}{dx}$$

$$2x + 1 = y + xy' + 3x^2y^4 + x^3 4y^3 y' \quad \Big| \frac{d}{dx}$$

$$2 = y' + 1y' + xy'' + 6xy^4 + 3x^2 4y^3 y' +$$

$$+ 3x^2 4y^3 y' + x^3 4\, 3y^2 y' y' + x^3 4y^3 y''$$

och dens sista ekvationen kan användas att räkna ut y''.

$$2 = y' + y' + y'' + 6 + 3 \cdot 4\, y' + 3 \cdot 4\, y' + 4{,}3\, y'^2$$
$$+ 4y''$$

$$2 = \underbrace{\boxed{}}_{\text{tal}} y' + \underbrace{\boxed{}}_{\text{tal}} y'' + \underbrace{\boxed{}}_{\text{tal}} y'^2$$

och vi vet y' från det tidigare steget, så att y'' kan räknas ut

$$\boxed{f''(x_1) \equiv y''}$$