MATEMATISK ANALYS, 2020 Mars 17 eftermiddags pass, facit

eftermiddags pass

(1) $z = \frac{z_1}{z_2}$

$z_1 = (2i^4 + 3i)(3 + 4i^3)$

$= \left\{ \begin{array}{l} i^4 = i^2 \cdot i^2 = (-1)\cdot(-1) = 1 \\ i^3 = i^2 i = -i \end{array} \right\}$

$= (2 + 3i)(3 - 4i)$

$= 6 - 8i + 9i - 12i^2 = \{ i^2 = -1 \}$

$= 6 - 8i + 9i + 12$

$= 18 + i$

$z_2 = (1+i)(2+i) = 2 + i + 2i + i^2 = \{ i^2 = -1 \}$

$= 2 + i + 2i - 1 = 1 + 3i$

$\frac{z_1}{z_2} = \frac{18 + i}{1 + 3i} = \{$ förläng med konjugat $\} =$

$= \frac{(18+i)(1-3i)}{(1+3i)(1-3i)} = \frac{18 - 3 \cdot 18 i + i - 3i^2}{1 + 3^2}$

$= \frac{18 + 3 - 54i + i}{10} = \frac{21 - 53i}{10}$

$$\boxed{\operatorname{Re} z = \frac{21}{10} \qquad \operatorname{Im} z = -\frac{53}{10}}$$

(2) $y = x^2 - 5x + \ln x$

ref : $x_1 = 1 \Rightarrow y_1 = 1 - 5 + \ln 1 = 1 - 5 + 0 = -4$

$x_2 = 1.1$, $y_2 = ?$

$\Delta x = x_2 - x_1 = 1.1 - 1 = 0.1$

$$\Delta y \approx y_2 - y_1 \approx \left.\frac{dy}{dx}\right|_{\text{ref.}} \cdot \Delta x = \left.\left(2x - 5 + \frac{1}{x}\right)\right|_{x=1} \cdot 0.1$$

$$= (2 - 5 + 1) \cdot 0.1$$

$$= -2 \times 0.1 = -0.2$$

$y_2 = y_1 + \Delta y = -4 - 0.2 = -4.2$

y minskar

(3)

$$f(x) = \begin{cases} g(x) & x \neq 1 \\ c & x = 1 \end{cases}$$

Notera att Df är inte hela R men bara ett interval, alltså [0,12/10]. Det är detta intervallet vi fokuserar på. Vad händer utanför intervallet, vi bryr oss inte om.

Kontinuitets villkor:

$$\lim_{x \to 1} f(x) = f(1)$$

$$\lim_{x \to 1} g(x) \qquad c$$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} \frac{x^3 - 2x^2 - x + 2}{x^4 - 7x + 6} = \left\{ \frac{1-2-1+2}{1-7+6} = \frac{0}{0} \right.$$

LH

$$= \lim_{x \to 1} \frac{3x^2 - 4x - 1}{4x^3 - 7} = \{ \text{BAMSE} \} = \frac{3-4-1}{4-7}$$

$$= \frac{2-4}{-3} = \frac{-2}{-3} = \frac{2}{3}$$

$\Rightarrow$ funktionen kan göras kontinuerlig i $x=1$ om man väljer $c = 1/2$

$\Rightarrow$ funktionen är kontinuerlig för (och med x i Df) alla $x \neq 1$. $g(x)$ är en rationell funktion och $g(x) \neq 0$ om $x \neq 1$ (enligt påståendet)

In intervallet det finns bara en farlig punkt, x=1, alla andra "faror" ligger utanför [0,12/10].

(4)

$$\lim_{x\to 0} \frac{\sin(3\sqrt{x})}{\sin\sqrt{3x}} = \left\{ \begin{array}{c} \frac{0}{0} \text{ form;} \\ \text{LH kan} \\ \text{användas} \end{array} \right\} =$$

$$= \lim_{x\to 0} \frac{\cos(3\sqrt{x})\cdot 3\left(\frac{1}{2\sqrt{x}}\right)}{\cos(\sqrt{3x})\cdot\sqrt{3}\left(\frac{1}{2\sqrt{x}}\right)}$$

$$= \lim_{x\to 0} \frac{3}{\sqrt{3}} \frac{\cos(3\sqrt{x})}{\cos(\sqrt{3x})} = \sqrt{3} \lim_{x\to 0} \frac{\cos(3\sqrt{x})}{\cos(\sqrt{3x})}$$

$$= \sqrt{3}\cdot 1 = \sqrt{3}$$

(5)

$$f(x) = x - 2\sqrt{x} + 2\ln(1+\sqrt{x})$$

$$f'(x) = 1 - 2\frac{1}{2\sqrt{x}} + 2\frac{1}{1+\sqrt{x}}\frac{1}{2\sqrt{x}}$$

$$= 1 + \frac{2}{2\sqrt{x}}\left(\frac{1}{1+\sqrt{x}} - 1\right)$$

$$= 1 + \frac{1}{\sqrt{x}}\frac{1-1-\sqrt{x}}{1+\sqrt{x}}$$

$$= 1 + \frac{1}{\sqrt{x}}\frac{-\sqrt{x}}{1+\sqrt{x}}$$

$$= 1 - \frac{1}{1+\sqrt{x}} = \frac{1+\sqrt{x}-1}{1+\sqrt{x}}$$

$$\boxed{f'(x) = \frac{\sqrt{x}}{1+\sqrt{x}}}$$

(6)

$$6x^2 = y + 2y^7 + 3y^{11} \quad \Big| \frac{d}{dx}$$

$$12x = y' + 14y^6 y' + 33y^{10} y'$$

$$12x = (1 + 14y^6 + 33y^{10}) \cdot y'$$

vāšena i $x = 1,\ y = 1$:

$$12 = (1 + 14 + 33)y'$$

$$y' = \frac{12}{15+33} = \frac{12}{48} = \frac{6}{24}$$

$$\boxed{y' = \frac{1}{4}}$$

7.

$$f(x) = \begin{cases} g_1(x) & x < 0 \\ g_2(x) & x \geq 0 \end{cases}$$

$g_1(x) = a \sin x + \cos x \cdot x$

$g_2(x) = \cos(x+b)$

Kontinuitet: $\lim_{x \to 0^-} g_1(x) = g_2(0)$

derivarbhet: $\lim_{x \to 0^-} g_1'(x) = \lim_{x \to 0^+} g_2'(x)$

ger två elevationer med två okända variabler; Kontinuitets villkor ger

$$a \cdot 0 + 0 = \cos b$$

och derivatan

$$g_1'(x) = a \cos x + \cos x - x \sin x$$

$$g_2'(x) = -\sin(x+b)$$

$\lim_{x \to 0^-} g_1'(x) = \lim_{x \to 0^+} g_2'(x)$ ger.

$$a + 1 = -\sin b$$

$$a = -1 - \sin b$$

välj $k = 0$

$$\boxed{\begin{aligned} 0 &= \cos b \\ a &= -1 - \sin b \end{aligned}} \Rightarrow \boxed{\begin{aligned} b &= \frac{\pi}{2} + k\pi \\ a &= -1 - 1 = -2 \end{aligned}}$$

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$$\int x^2 \ln x \, dx = \left\{ \begin{array}{l} u = x^2 \\ dv = \ln x \, dx \end{array} \right\}$$

funkar inte

$$= \left\{ \begin{array}{ll} dv = x^2 dx & \rightarrow v = x^3/3 \\ u = \ln x & \rightarrow du = \frac{1}{x} dx \end{array} \right\}$$

$$= uv - \int v \, du = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \frac{x^3}{3} + c$$

$$= -\frac{x^3}{9} + \frac{x^3 \ln x}{3} + c$$

(9) $\int \frac{x^4+1}{x^2-4}dx$ är en obestämd integral.

grad $(x^4+1) = 4$

grad $(x^2-4) = 2$

grad(täljare) > grad(nämnare)

→ reducera med polynom division

$$(x^4+1):(x^2-4) = x^2+4$$

$$\begin{array}{l} x^4 - 4x^2 \\ \hline 0 + 4x^2 + 1 \\ \quad 4x^2 - 16 \\ \hline \quad\quad 17 \end{array}$$

$$\frac{x^4+1}{x^2-4} = x^2+4+\frac{17}{x^2-4}$$

$$\int \frac{x^4+1}{x^2-4}dx = \int \left\{ x^2+4+\frac{17}{x^2-4} \right\} dx$$

$$= \frac{x^3}{3} + 4x + 17\int \frac{dx}{x^2-4}$$

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2)$$

$$1 = (A+B)x + 2(A-B)$$

$$A+B=0 \quad \Rightarrow \quad A = \frac{1}{4}$$

$$A-B = \frac{1}{2} \qquad B = -\frac{1}{4}$$

Kontrollar:

$$\frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right) = \frac{1}{4}\,\frac{x+2-x+2}{(x-2)(x+2)}$$

$$= \frac{1}{4}\cdot\frac{2\cdot 2}{x^2-4} \checkmark$$

$$\int \frac{x^4+1}{x^2-4}\,dx = \frac{x^3}{3} + 4x + \frac{17}{4}\left\{\int \frac{dx}{x-2} - \int \frac{dx}{x+2}\right\}$$

$$= \frac{x^3}{3} + 4x + \frac{17}{4}\left\{\ln|x-2| - \ln|x+2|\right\}$$

$$= \frac{x^3}{3} + 4x + \frac{17}{4}\ln\left|\frac{x-2}{x+2}\right| + c$$

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$$\tan x \sin y \, dx = -\cos x \cot y \, dy$$

$$\frac{\tan x}{\cos x} dx = -\frac{\cot y}{\sin y} dy$$

$$\frac{\sin x}{\cos^2 x} dx = -\frac{\cos y}{\sin^2 y} dy \quad \Big| \int$$

$$\int \frac{\sin x}{\cos^2 x} dx = -\int \frac{\cos y}{\sin^2 y} dy$$

$$\left(\frac{1}{\cos x}\right)'_x = -\frac{1}{\cos^2 x}(-)\sin x = \frac{\sin x}{\cos^2 x}$$

$$\left(\frac{1}{\sin y}\right)'_y = -\frac{1}{\sin^2 y} \cdot \cos y = -\frac{\cos y}{\sin^2 y}$$

$$\frac{1}{\cos x} = \frac{1}{\sin y} - c$$

$x = 0$, $\cos x = 1$

$y = \frac{\pi}{6}$, $\sin\frac{\pi}{6} = \frac{1}{2}$

$$\frac{1}{\sin y} = \frac{1}{\cos x} + c$$

$$\frac{1}{\frac{1}{2}} = \frac{1}{1} + c$$

$2 = 1 + c$

$\Downarrow$

$c = 1$

$$\sin y = \frac{1}{\frac{1}{\cos x} + c} = \frac{\cos x}{1 + c\cos x}$$

$$y = \arcsin\left(\frac{\cos x}{1 + c\cos x}\right)$$

$$y = \arcsin\left(\frac{\cos x}{1 + \cos x}\right)$$

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$$y'' - 2y' + 5y = 1$$

homogen:

$$y_h'' - 2y_h' + 5y_h = 0 \qquad y_h = e^{\lambda x}$$

$$\lambda^2 - 2\lambda + 5 = 0 \implies \lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4\cdot 1\cdot 5}}{2}$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$\implies 1 \pm 2i$$

$$y_h(x) = e^x \left(A \sin 2x + B \cos 2x \right)$$

icke-homogen:

$$y_p'' - 2y_p' + 5y_p = 1 \qquad y_p = c$$

$$y_p' = 0$$

$$0 - 2\cdot 0 + 5\cdot c = 1 \qquad y_p'' = 0$$

$$\implies c = 1/5$$

$$y(x) = y_h(x) + y_p(x) = e^x\left(A \sin 2x + B \cos 2x\right) + \frac{1}{5}$$

$$0 = y(0) = (A\cdot 0 + B\cdot 1) + \frac{1}{5} \implies B = -\frac{1}{5}$$

$$y'(x) = e^x(A \sin 2x + B\cos 2x) +$$
$$+ e^x 2(A \cos 2x - B \sin 2x)$$

$$\frac{1}{5} = y'(0) = A\cdot 0 + B\cdot 1 + 2(A\cdot 1 - B\cdot 0) = 2A + B$$

$B = -\frac{1}{5}$

$2A + B = \frac{1}{5}$

$2A = \frac{1}{5} - B = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

$A = \frac{1}{5}$

$$y(x) = e^{x}\left(\frac{\sin 2x}{5} - \frac{\cos 2x}{5} \right) + \frac{1}{5}$$

$$\boxed{y(x) = \frac{1}{5}\left\{ 1 + e^{x}\left(\sin 2x - \cos 2x \right) \right\}}$$

(12)

$$\lim_{x\to 0} (\cot\sqrt{x})^{\frac{1}{\ln x}} = \left\{ \text{form } \infty^{\frac{1}{\infty}} \right\}$$

$$= \lim_{x\to 0} e^{\frac{1}{\ln x}\ln\cot\sqrt{x}}$$

$$= e^{\alpha}, \quad \alpha = \lim_{x\to 0} \frac{\ln\cot\sqrt{x}}{\ln x} = \left\{ \frac{\infty}{\infty} \right\}$$

$$\alpha \overset{LH}{=} \lim_{x\to 0} \frac{\frac{(\cot\sqrt{x})'}{\cot\sqrt{x}}}{\frac{1}{x}} = \lim_{x\to 0} x \tan\sqrt{x}\,(\cot\sqrt{x})'$$

$$(\cot\sqrt{x})' = (\cot u)' \cdot \frac{1}{2\sqrt{x}} = \left(\frac{\cos u}{\sin u}\right)'_u \frac{1}{2\sqrt{x}}$$

$$= \frac{-\sin^2 u - \cos^2 u}{\sin^2 u} \frac{1}{2\sqrt{x}}$$

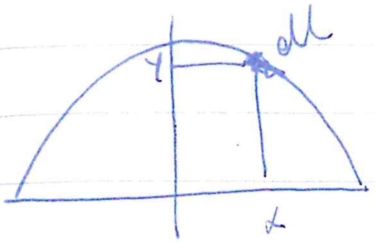
$$= -\frac{1}{\sin^2 u} \frac{1}{2\sqrt{x}}$$

$$\alpha = \lim_{x\to 0} x \tan\sqrt{x} \frac{1}{\sin^2\sqrt{x}} (-) \frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{\sqrt{x}\,\cancel{\sin\sqrt{x}}}{\cos\sqrt{x}\cdot \sin^{\cancel{2}}\sqrt{x}} =$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{1}{\cos\sqrt{x}} \cdot \lim_{x\to 0} \frac{\sqrt{x}}{\sin\sqrt{x}} = -\frac{1}{2}\cdot 1\cdot 1$$

(13)



$$x_{cm} = \int x\, dl \cdot \sigma \Big/ \int dl \cdot \sigma$$

$$y_{cm} = \int y\, dl \cdot \sigma \Big/ \int dl \cdot \sigma$$

$$x_{cm} = \frac{\int_{-3}^{3} x\, dl}{\int_{-3}^{3} dl} \qquad y_{cm} = \frac{\int_{-3}^{3} y\, dl}{\int_{-3}^{3} dl}$$

$$y^2 + x^2 = 3^2 \qquad y = \sqrt{3^2 - x^2} \qquad y' = \frac{1}{2\sqrt{9-x^2}}(-2x)$$

$$dl = \sqrt{1 + y'^2}\, dx = \sqrt{1 + \frac{x^2}{9-x^2}}\, dx \qquad y' = -\frac{x}{\sqrt{9-x^2}}$$

$$= \sqrt{\frac{9 - x^2 + x^2}{9-x^2}}\, dx = \frac{3}{\sqrt{9-x^2}}\, dx$$

$$\int_{-3}^{3} x\, dl = \int_{-3}^{3} x \cdot \frac{3}{\sqrt{9-x^2}}\, dx = 0 \quad \text{för att}$$

vi integrerar en funktion som är udda med symetriska gränser.

$$\int_{-3}^{3} y\, dl = \int_{-3}^{3} \sqrt{9-x^2} \cdot \frac{3}{\sqrt{9-x^2}}\, dx$$

$$= \int_{-3}^{3} 3\, dx = 3\, x\Big|_{-3}^{3} = 3(3+3) = 3 \cdot 2 \cdot 3$$

$$= 18$$

$$\int_{-3}^{3} dl = \int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\, dx = \left\{ \begin{array}{l} x = 3u \\ dx = 3\,du \end{array} \right\}$$

$$= \int_{-1}^{1} \frac{3 \cdot 3 \cdot du}{\sqrt{9-9u^2}} = \frac{3 \cdot 3}{\sqrt{9}} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{3 \cdot \cancel{3}}{\cancel{3}} \cdot 2 \underbrace{\int_{0}^{1} \frac{du}{\sqrt{1-u^2}}}_{\arcsin(1) - \arcsin 0}$$

$$= 6\left(\frac{\pi}{2} - 0\right) = 3\pi$$

$$\boxed{\begin{array}{l} x_{cm} = 0 \\ y_{cm} = \dfrac{18}{3\pi} = \dfrac{6}{\pi} \end{array}}$$