Matematisk analys, Tenta, facit, Mars 19, 2020

(Förmiddags pass)

(1)
$$z = \frac{(3i^4+2i)(4+3i^5)}{(1+i)(2-i)} = \left\{ \begin{array}{l} i^4 = (i^2)^2 = (-1)^2 = 1 \\ i^5 = i^4 \cdot i = i \end{array} \right\} =$$

$$= \frac{(3+2i)(4+3i)}{(1+i)(2-i)} = \frac{12+9i+8i+6i^2}{2-i+2i-i^2} = \{ i^2 = -1$$

$$= \frac{12+9i+8i-6}{2-i+2i-(-1)} = \frac{6+17i}{3+i} = \{ \text{förlänga med konjugat}$$

$$= \frac{(6+17i)(3-i)}{(3+i)(3-i)} = \frac{18-6i+17\cdot 3i-17i^2}{9+1}$$

$$= \frac{18-17(-1)+(17\cdot 3-2\cdot 3)i}{10} = \frac{18+17+(17-2)\cdot}{10}$$

$$= \frac{35+15\cdot 3i}{2\cdot 5} = \frac{7\cdot \cancel{5} + \overset{3\cdot\cancel{5}}{15}\cdot 3i}{2\cdot \cancel{5}}$$

$$= \frac{7+9i}{2} = \frac{7}{2} + \frac{9}{2}i$$

$$\operatorname{Re} z = \frac{7}{2}, \quad \operatorname{Im} z = \frac{9}{2}$$

(2) $y = 2x^3 - 5x^2 + \sin(x-1)$

$x_1 = 1$ (referens) $\Rightarrow$ $y_1 = 2\cdot1^3 - 5\cdot1^2 + \sin(0)$

$= 2-5 = -3$

$x_2 = 1.1 \Rightarrow y_2 = ?$

$\Delta x = x_2 - x_1 = 1.1 - 1 = 0.1$

$\Delta y = y_2 - y_1 \approx \left(\frac{dy}{dx}\right)_{ref} \cdot \Delta x$

$= \left[6x^2 - 10x + \cos(x-1)\right]_{x=1} \cdot 0.1$

$= \left[6 - 10 + \cos 0\right] \cdot 0.1$

$= (-4+1)\,0.1 = -3 \times 0.1 = -0.3$

$y_2 = y_1 + \Delta y = -3 - 0.3 = -3.3$

y minskar

③ $f(x) = \begin{cases} g(x) & x \neq 1 \\ c & x = 1 \end{cases}$

$\boxed{\text{SVAR: Ja, välj } c = \frac{2}{3}}$

$$g(x) = \frac{x^4 - 2x^3 - 2x + 3}{x^6 - 6x^2 + 5}$$

$D_f = [0,12/10]$
Det är detta intervallet vi fokuserar på och ignorerar allt som händer utanför!

försök med "BAMSE": $g(1) = \frac{1-2-2+3}{1-6+5} = \frac{0}{0}$

(1) $g(x)$ är kontinuerlig för alla $x \neq 1$ för att den är definierat som en rationell funktion, som är kontinuerlig i alla punkter där nämnaren är inte noll. g(x) har noll punkter utanför Df, men dessa kan ignoreras. Bara noll punkter som ligger i Df är viktiga, x=1 är den enda.

(2) $x=1$ är den "problematiska punkten". vi kan göra $f(x)$ kontinuerlig om

$$\lim_{x \to 1} f(x) = f(1)$$

$$\lim_{x \to 1} g(x) \qquad c$$

alltså, $f(x)$ blir kontinuerlig om vi väljer c sådant att

$$c = \lim_{x \to 1} g(x)$$

$$\lim_{x \to 1} g(x) = \left\{ \frac{0}{0} \text{ formen} \right\} \overset{LH}{=} \lim_{x \to 1} \frac{4x^3 - 6x^2 - 2}{6x^5 - 12x}$$

$$\text{BAMSE} \left\{ = \frac{4-6-2}{6-12} = \frac{-4}{-6} = \frac{2}{3} \right.$$

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$$\lim_{x\to 1} \frac{\sin(3\sqrt{x-1})}{2\sqrt{x-1}} = \left\{ \frac{0}{0} \right\} \overset{LH}{=}$$

$$= \lim_{x\to 1} \frac{\cos(3\sqrt{x-1})\cdot 3 \cdot \frac{1}{2\sqrt{x-1}}}{2\cdot \frac{1}{2\sqrt{x-1}}}$$

$$= \frac{3}{2} \lim_{x\to 1} \cos(3\sqrt{x-1}) = \frac{3}{2}$$

Ett annat sätt är att använda variabel byte $u = x-1$

$$\lim_{x\to 1} \frac{\sin(3\sqrt{x-1})}{2\sqrt{x-1}} = \left\{ \begin{matrix} u = x-1 \\ x\to 1 \\ u \to 0 \end{matrix} \right\}$$

$$= \lim_{u\to 0} \frac{\sin(3\sqrt{u})}{2\sqrt{u}} = \left\{ \begin{matrix} u = w^2 \\ u\to 0 \\ w\to 0 \end{matrix} \right\}$$

$$= \lim_{w\to 0} \frac{\sin(3\cdot w)}{2\cdot w} = \left\{ \begin{matrix} 3w = \alpha \\ w\to 0 \\ \alpha\to 0 \end{matrix} \right\}$$

$$= \lim_{\alpha\to 0} \frac{\sin\alpha}{2\cdot \frac{\alpha}{3}} = \frac{3}{2} \underbrace{\lim_{\alpha\to 0} \frac{\sin\alpha}{\alpha}}_{\text{ganska känd gränsvärdet}} = \frac{3}{2}$$

(5) $f(x) = \ln^2 x - \ln(\ln x)$

$$f'(x) = 2\ln x\,(\ln x)' - \frac{1}{\ln x}\cdot(\ln x)'$$

$$= \left(2\ln x - \frac{1}{\ln x}\right)(\ln x)'$$

$$= \frac{2\ln^2 x - 1}{\ln x}\cdot\frac{1}{x}$$

$$= \frac{2\ln^2 x - 1}{x\cdot\ln x}$$

(6) $2x^2 + 7 = y + 3y^4 + 5y^5 \quad \Big| \frac{d}{dx}$

$$4x = y' + 12y^3y' + 25y^4y'$$

$$4x = (1 + 12y^3 + 25y^4)\,y'$$

$x = 1,\ y = 1$ ger

$$4 = (1 + 12 + 25)\,y'$$

$$y' = \frac{4}{38} = \frac{2}{19}$$

(7.) $f(x) = \begin{cases} x^2 + ax + \cos x & x < 0 \\ \sin(x+b) & x \geq 0 \end{cases}$

Två villkor som garanterar deriverbarhet är

(1) $f(x)$ måste vara kontinuerlig i $x = 0$, då är det sant att, t.ex, $f'_{HÖ}(0) = \lim_{x \to 0^+} f'(x)$

(2) $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x)$

Kontinuitet:

$\lim_{x \to 0} f(x) = f(0) = \sin(0+b)$

$\lim_{x \to 0} f(x)$ bestäms av

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

samma

$0^2 + a \cdot 0 + \cos 0 = \sin(0+b) \Rightarrow \boxed{1 = \sin b}$

der. $\lim_{x \to 0} (2x + a - \sin x) = \lim_{x \to 0} (\cos(x+b))$

$\Downarrow$

$\boxed{a = \cos b}$

$\Rightarrow$ $b = \frac{\pi}{2} + 2k\pi \quad k = 0, \pm 1, \ldots$

$a = \cos b = \cos(\frac{\pi}{2} + 2k\pi) = \cos \frac{\pi}{2} = 0$

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sätt 1: $\int x^2 \sin x\, dx = \left\{ \begin{array}{l} u = x^2;\ du = 2x\,dx \\ dv = \sin x\, dx \\ v = -\cos x \end{array} \right\}$

$= uv - \int v\, du = x^2(-\cos x) - \int (-\cos x) 2x\, dx$

$= -x^2 \cos x + 2\int x \cos x\, dx$

en gång till:

$\int x \cos x\, dx = \left\{ \begin{array}{ll} u = x & \Rightarrow\ du = dx \\ dv = \cos x\, dx & \quad v = \sin x \end{array} \right\}$

$= uv - \int v\, du = x \cdot \sin x - \int \sin x\, dx$

$= x \sin x - (-\cos x)$

$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x$

$= 2x \sin x + (2 - x^2)\cos x$

sätt 2: $\int x^2 \sin x\, dx = \left\{ \begin{array}{ll} u = \sin x & \Rightarrow\ du = \cos x\, dx \\ dv = x^2 dx & \quad v = x^3/3 \end{array} \right\}$

$= uv - \int v\, du = \frac{x^3}{3} \sin x - \int \frac{x^3}{3} \cos x\, dx$

detta blir inte bra för att potensen i x ökar. Integralen blir bara svårare.

9. $\int \frac{x^4-78}{x^2-9}\,dx$ är en obestämd integral

grad täljare > grad nämnare → reducering

$(x^4-78):(x^2-9) = x^2+9$

x^4-9x^2

$0+9x^2-78$

$9x^2-81$

$0-78+81=3$

$$\Rightarrow \quad \frac{x^4-78}{x^2-9} = x^2+9+\frac{3}{x^2-9}$$

$$\int \frac{x^4-78}{x^2-9}\,dx = \int\left(x^2+9+\frac{3}{x^2-9}\right)dx$$

$$= \frac{x^3}{3}+9x+3\int\frac{dx}{x^2-9}$$

$$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3}+\frac{B}{x+3}$$

$$1 = A(x+3)+B(x-3) = (A+B)x+3(A-B)$$

$A+B=0 \qquad A=\frac{1}{6}$

$A-B=1/3 \qquad B=-\frac{1}{6}$

$$\frac{1}{x^2-9} = \frac{\frac{1}{6}}{x-3} + \frac{\left(-\frac{1}{6}\right)}{x+3} = \frac{1}{6}\left(\frac{1}{x-3} - \frac{1}{x+3}\right)$$

kontrol : $\frac{1}{6}\left(\frac{1}{x-3} - \frac{1}{x+3}\right) = \frac{1}{6} \frac{\cancel{x}+3-\cancel{x}+3}{(x-3)(x+3)}$

$$= \frac{1}{6}\frac{2\cdot 3}{x^2-9} = \frac{1}{x^2-9} \quad \text{ok}$$

$$\int \frac{x^4-78}{x^2-9}\,dx = \frac{x^3}{3} + 9x + 3\cdot\frac{1}{6}\int\left(\frac{1}{x-3} - \frac{1}{x+3}\right)dx$$

$$= \frac{x^3}{3} + 9x + \frac{1}{2}\left\{\int\frac{dx}{x-3} - \int\frac{dx}{x+3}\right\}$$

$$= \frac{x^3}{3} + 9x + \frac{1}{2}\left\{\ln|x-3| - \ln|x+3|\right\} + c$$

$$= \frac{x^3}{3} + 9x + \frac{1}{2}\ln\left|\frac{x-3}{x+3}\right| + c$$

(10.) $(1+e^x)\,y\,dy - e^x\,dx = 0$

$$(1+e^x)\,y\,dy = e^x\,dx$$

$$y\,dy = \frac{e^x}{1+e^x}\,dx \quad \Big| \int$$

$$\int y\,dy = \int \frac{e^x}{1+e^x}\,dx$$

$$\frac{y^2}{2} = \ln|1+e^x| + \frac{c}{2}$$

$1+e^x > 0 \Rightarrow |1+e^x| = (1+e^x)$

$$\frac{y^2}{2} = \ln(1+e^x) + \frac{c}{2}$$

$$y = \pm\sqrt{c + 2\ln(1+e^x)}$$

$y(0) = 1$

$$\Rightarrow 1 = \pm\sqrt{c + 2\ln 2}$$

man måste välja den positiva roten

$1 = \sqrt{c + 2\ln 2}$, $1 = c + 2\ln 2$

$c = 1 - 2\ln 2$

$$\boxed{y(x) = +\sqrt{1 - 2\ln 2 + 2\ln(1+e^x)}}$$

(11) $y'' - 2y' + 2y = x$

homogen eleavation:

$$y'' - 2y' + 2y = 0 \qquad y = e^{\lambda x}$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda^2 - 2\lambda + 1 - 1 + 2 = 0$$

$$(\lambda - 1)^2 = -1 \quad , \quad \lambda = 1 \pm i$$

$$y_h(x) = A e^x \sin x + B e^x \cos x$$

partikulär lösning $y_p = ax + b$

$$y_p' = a \qquad 0 - 2 \cdot a + 2 \cdot (ax + b) = x$$

$$y_p'' = 0 \qquad \underbrace{2a}_{1} x + 2\underbrace{(b-a)}_{0} = x$$

$$a = \frac{1}{2} \qquad b = a = \frac{1}{2}$$

$$y(x) = y_h(x) + y_p(x) = e^x(A \sin x + B \cos x) + \frac{x+1}{2}$$

$$0 = y(0) = B + \frac{1}{2} \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$1 = y'(0) = \left\{ e^x(\dots) + e^x(A \cos x - B \sin x) + \frac{1}{2} \right\}_{x=0}$$

$$= 1(B \cdot 1) + 1(A \cdot 1) + \frac{1}{2}$$

$$1 = \underbrace{B + \frac{1}{2}}_{0} + A \Rightarrow \boxed{A = 1}$$

(12)

$$\lim_{x\to 0} \left(\cos(2\sqrt{x})\right)^{\frac{3}{x}} = \lim_{x\to 0} e^{\ln\left(\cos(2\sqrt{x})\right)^{\frac{3}{x}}}$$

$$= \lim_{x\to 0} e^{\underbrace{\frac{3}{x}\ln\cos(2\sqrt{x})}_{\frac{0}{0}\text{ form}}}$$

$$= e^{\lim_{x\to 0} \frac{3\ln\cos(2\sqrt{x})}{x}}$$

$$\overset{LH}{=} e^{3\lim_{x\to 0} \frac{\frac{1}{\cos 2\sqrt{x}}\cdot(-)\sin(2\sqrt{x})\cdot \cancel{2}\frac{1}{\cancel{2}\sqrt{x}}}{1}}$$

$$= e^{-3\lim_{x\to 0} \frac{\sin(2\sqrt{x})}{\cos(2\sqrt{x})\cdot\sqrt{x}}}$$

$$= e^{-3 \underbrace{\lim_{x\to 0}\frac{1}{\cos(2\sqrt{x})}}_{1} \cdot \underbrace{\lim_{x\to 0}\frac{\sin 2\sqrt{x}}{\sqrt{x}}}_{=2}}$$

(variable type alex LH)

$$= e^{-3\cdot 1\cdot 2} = e^{-6}$$

$$= \frac{1}{e^6}$$

x - led

y - led

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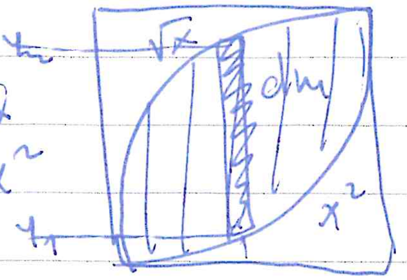
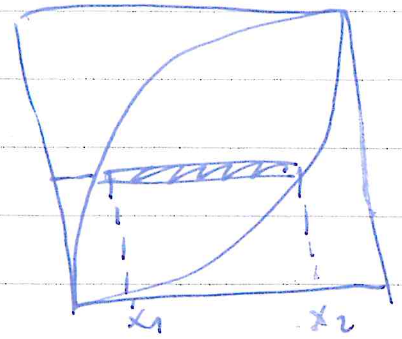
$y_2 = \sqrt{x}$

$y_1 = x^2$





$dm = (y_2 - y_1) \cdot \sigma \cdot dx \overset{x}{=} (\sqrt{x} - x^2) \cdot \sigma \, dx$

$x_{cm} = \dfrac{I_{x1}}{I_{x0}}$

$I_{x1} = \int_0^1 x(\sqrt{x} - x^2)\, dx$

$I_{x0} = \int_0^1 (\sqrt{x} - x^2)\, dx$

$dm = (x_2 - x_1) \cdot \sigma \cdot dx$

$y = \sqrt{x_1} = x_2^2$

$y_{cm} = \dfrac{I_{y1}}{I_{y0}}$

$I_{y1} = \int_0^1 y(x_2 - x_1)\, dy$

$= \int_0^1 y(\sqrt{y} - y^2)\, dy$

$I_{y0} = \int_0^1 (\sqrt{y} - y^2)\, dy$

$x \Leftrightarrow y$

$dx = dy$

med enkelt variabel byte man kan se at $x_{cm} = y_{cm}$

$I_{x1} = \int_0^1 (x^{3/2} - x^3)\, dx = \left(\dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} - \dfrac{x^4}{4} \right) \Big|_0^1 =$

$= \left(\frac{2}{5} x^{\frac{5}{2}} - \frac{1}{4} x^4 \right) \Big|_0^1 = \frac{2}{5} - \frac{1}{4} = \frac{8-5}{20} = \frac{3}{20}$

$I_{x0} = \left(\dfrac{x^{3/2}}{3/2} - \dfrac{x^3}{3} \right) \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$\boxed{x_{cm} = y_{cm} = \dfrac{\frac{3}{20}}{\frac{1}{3}} = \dfrac{9}{20}}$