EXAMINATION: Tentamensskrivning i Matematisk Statistik (TMS061)
(TMS060)

Time: Thursday 30 August 2007

Jour: Viktor Olsbo, 0730 - 888826

Aid: You are allowed to use a scientific calculator and a half page (both sides) of hand written notes

Lab: Depending on the performance in the lab 0-5 points will be added to your test score to provide with your final score.

Grade: You need points for 29 for 5, 22 points for 4 and 15 points for 3.

Motivate all your answers. Good Luck!

1) a) $P(B|A) = P(B|A') = 0.2$. Compute $P(B)$. (A' = complement to A).

b) $Var(X) = 1.5$. Compute $Var(2 - X)$.

c) X is $Po(\lambda)$. Explain why $2X$ cannot be $Po(2\lambda)$. (6p)

2) Let $X_1, X_2, \ldots, X_n$ be a random sample.

a) What conditions do $X_1, X_2, \ldots, X_n$ have to satisfy?

b) If $X_1, X_2, \ldots, X_n$ are additionally normally distributed $N(\mu, \sigma^2)$ what is the distribution of $\bar{X}$?

c) Find $E(\bar{X})$, $Var(\bar{X})$?

d) State the Central Limit Theorem.

(5p)

3) X is a geometric random variable with parameter p if $P(X = x) = p(1-p)^{x-1}, x = 1, 2, 3, \ldots$. For which p is $P(X \leq 2) > 0.75$? (4p).

4) Consider the following frequency table:

Values	0	1	2	3	4	5
Observed frequency	75	140	108	66	9	2

Based on 400 observations is a binomial distribution $B(5, 0.3)$ an appropriate model? Perform a χ^2 test with $\alpha = 0.05$. (5p)

5) Let X be a random variable with probability density function

$$P(X = x) = (x-1)p^2(1-p)^{x-2}, x = 2, 3, \ldots$$

Find the maximum likelihood estimator of p. (5p)

6) The probability density function for the time X it takes to assist a customer is $f(x) = \frac{10-x}{50}, 0 < x < 10$.

a) Compute $E(X)$ and $Var(X)$.

b) For another customer Y the time is $8 + X$. Compute $E(Y)$ and $Var(Y)$.

(4p)

7) Two different materials can be used for a mechanical construction. To decide which material to use five samples from each material have been tested. The results follow:

Material 1: $5.60, 9.92, 6.03, 5.53, 7.30$
Material 2: $4.26, 4.47, 6.79, 6.20, 6.61$

Provide with a confidence interval for the difference between the two materials at a confidence level $\alpha = 0.05$. State clearly all the necessary assumptions.

(4p)

8) Six tests for one type of material gave (in kp/mm^2)

$$6.7, 7.5, 7.2, 7.3, 6.9, 7.6$$

Suppose that the sample is from a normal population with mean value μ. Test $H_0 : \mu = 7$ against $H_1 : \mu > 7$ at significant level $\alpha = 0.05$. (3p)

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1a) $P(B) = P(B|A)\cdot P(A) + P(B|A')\cdot P(A') =$

$= P(B|A)\cdot P(A) + P(B|A)\cdot P(A') =$

$= P(B|A)\cdot\left(P(A)+P(A')\right) =$

$= P(B|A)\cdot 1 = \underline{\underline{0.2}}$ 1p

1b) $Var(2-X) = Var X = 15$ 1p

1c) If $X \sim Po(\lambda) \Rightarrow Var X = \lambda$.

But $Var(2X) = 4\cdot Var X = 4\lambda \neq$ Var of a $Po(2\lambda)$. 2p

<u>OR</u> $E(2X) = 2\lambda \neq 4\lambda = Var(2X)$

2 a) idependent and identically distributed 1

b) normal 1

c) $E\overline{X} = E\left(\frac{X_1+\cdot\ +X_n}{n}\right) = \frac{n\cdot\mu}{n} = \mu$. 1

$Var\overline{X} = Var\left(\frac{X_1+\cdot\ +X_n}{n}\right) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

~~3) $P(X\geq 2) = 1 - P(X<2) = 1 - P(X=0) -$~~

~~$- P(X=1) = 1 - p(1-p) = 1 - p = 0.75 \Rightarrow$~~

~~$p = 0.25$~~

3a) $P(X \leq 2) = P(X=1) + P(X=2) =$

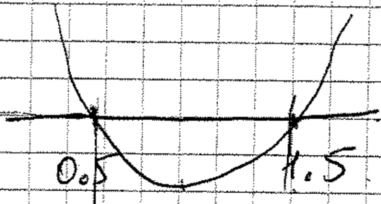
$p + p(1-p) = p \cdot (1+1-p) = p(2-p) = 2p - p^2 \geq 0.75$

$\Rightarrow p^2 - 2p + 0.75 \leq 0$

$\Delta = 4 - 4 \cdot 0.75 = 4 - 3.00 = 1 > 0$

$p_{1,2} = \frac{2 \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 1}{2} = \begin{cases} 1.5 \\ 0.5 \end{cases}$

3p



So $p \in [0.5, 1]$

4) $X \sim B(n,p) \Rightarrow P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$P(X=0) = 0.1681$ $\quad E_1 = 67.228$

$P(X=1) = 0.3601$ $\quad E_2 = 144.06$

$P(X=2) = 0.3087$ $\quad E_3 = 123.48$

$P(X=3) = 0.1323$ $\quad E_4 = 52.92$

$P(X=4) = 0.0283$ $\quad E_5 = 11.34$

$P(X=5) = 0.0024$ $\quad E_6 = 0.9720 < 3$

5p

$\tilde{P}_5 = 0.0283 + 0.0024$ $\quad \tilde{E}_5 = 12.28$

$\chi_0^2 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = 6.3194$, $k - p - 1 = 5 - 0 - 1 = 4$ df

$\chi^2_{0.05,4} = 9.49$

Since $\chi_0^2 = 6.3194 < 9.49$ <u>fail</u> to reject H_0

5) $L(p) = \prod_{i=1}^{n} (x_i - 1) \cdot p^2 \cdot (1-p)^{x_i - 2} =$

$= \cancel{p^{2n}}\ p^{2n} \prod_{i=1}^{n} (x_i - 1)(1-p)^{x_i - 2}$

5

$\log L(p) = 2n \log p + \sum_{i=1}^{n} \log (x_i - 1) + \sum_{i=1}^{n} (x_i - 2) \log (1-$

$\frac{\partial \log L(p)}{\partial p} = \frac{2n}{p} + \sum_{i=1}^{n} \frac{-(x_i - 2)}{1-p} = \frac{2n}{p} - \sum_{i=1}^{n} \frac{(x_i - 2)}{1-p} =$

$= 0 \Rightarrow \frac{2n}{p} = \sum_{i=1}^{n} \frac{x_i - 2}{1-p} \Rightarrow$

$\frac{1-p}{p} = \sum_{i=1}^{n} \frac{x_i - 2}{2n} \Rightarrow \frac{1}{p} - 1 = \cancel{\frac{1}{2n}} \frac{1}{2} \bar{X} - \frac{2n}{2n} \Rightarrow$

$\frac{1}{p} - 1 = \frac{\bar{X}}{2} - 1 \Rightarrow \hat{p} = \frac{2}{\bar{X}}$

6) $f(x) = \frac{10 - x}{50}, \quad x \in (0, 10).$

$EX = \int_0^{10} x f(x)\, dx = \int_0^{10} x \cdot \frac{10 - x}{50}\, dx =$

$= \frac{1}{50} \int_0^{10} 10x - x^2\, dx = \frac{1}{50} \cdot \frac{10x^2}{2} - \frac{x^3}{3} \Big/_0^{10} =$

$= \frac{1}{50} \left(5 \cdot (100 - 0) - \frac{100 \cdot 10}{3} - 0 \right) = \frac{1}{50} \cdot \left(500 - \frac{1000}{3} \right) = \frac{10}{3}$

$$EX^2 = \int_0^{10} x^2 \cdot \frac{10-x}{50}\, dx = \frac{1}{50} \int_0^{10} 10x^2 - x^3\, dx =$$

$$= \frac{1}{50} \left. \frac{10x^3}{3} - \frac{x^4}{4} \right|_0^{10} = \frac{1}{50}\left(\frac{10.1000}{3} - \frac{10000}{4}\right) =$$

$$= \frac{50}{3}.$$

$$Var\,X = EX^2 - (EX)^2 = \frac{50}{3} - \frac{100}{9} = \frac{50}{9}.$$

b) $EY = E(8+X) = 8 + EX = 8 + \frac{10}{3} = \frac{34}{3}.$

$Var\,Y = Var(8+X) = Var\,X = \frac{50}{9}.$

7)

Assuming the two samples are independent from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$.

$$\mu_1 - \mu_2 = \bar{X}_1 - \bar{X}_2 \pm t_{0.025}^{(8)} \cdot sp \sqrt{\frac{1}{5} + \frac{1}{5}} =$$

$$= 6.876 - 5.666 \pm 2.306 \cdot \sqrt{2.431} \cdot \sqrt{\frac{1}{5} + \frac{1}{5}} =$$

$$= \underline{\underline{1.21 \pm 2.27}}.$$

8) $\bar{X} = 7.2$

$s = 0.3464$

$$t_{obs} = \frac{7.2 - 7}{\frac{0.3464}{\sqrt{6}}} = 1.41$$

$t_{5,0.05} = 2.015 > t_{obs}$ Cannot reject H_0 at $\alpha = 0.05$.

3d) If $X_1, \ldots, X_n$ is a random sample of size n taken from a pop with mean μ and finite variance σ^2, and if $\bar{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}, \text{ as } n \to \infty \text{ is the}$$

standard normal. (2p)