1. (a) False. The three closest neighbors to $(0.1, 0.3)$ are all of class 2.
   (b) False. The semivariogram is defined as $\gamma(\mathbf{s}_1, \mathbf{s}_2) = 0.5\mathsf{V}(X(\mathbf{s}_1) - X(\mathbf{s}_2))$.
   (c) False.
   (d) True.
   (e) True.

2. (a) If we let $N(A)$ denote the number of points in the region A, with area $|A|$, the probability is
$$\mathsf{P}(N(A) = k) = e^{-\lambda|A|}\frac{(\lambda|A|)^k}{k!}.$$
   (b) Let λ be the intensity of the point process and let $B_r(x)$ denote a ball of radius r centered at a point x in the point pattern X. The K function is defined as
$$K(r) = \frac{1}{\lambda}\mathsf{E}(N(B_r(x)) - 1|x \in X).$$
   (c) Since the K-function counts the number of additional points in a ball with radius r around a location in the point pattern, a reasonable estimate it to take the average number of points in the balls $B_r(x)$ centered in each location $x \in X$. To get an unbiased estimator, we should take into account that there may be other locations outside the observation window that we have not observed. Therefore, an estimator is
$$\hat{K}(r) = \frac{1}{\hat{\lambda}^2|D_r|}\sum_{x\in D_r\cap X}\sum_{y\in X} 1(\leq \|x-y\| \leq r),$$
   where D_r is the set of locations which are at least a distance r away from the boundary of D, N_r, and $\hat{\lambda} = |X|/|D|$ is an estimate of the intensity of the point process.
   (d) Since the estimated K-function is above the K-function for the homogeneous Poisson process, the model does not seem to fit the data, and we should instead use a clustering process.

3. (a) We can use image filtering with a vertical edge filter: We obtain a filtered image by computing the convolution between the image and a filter kernel w. For the vertical edge filter we have
$$\begin{pmatrix} w_{-1,-1} & w_{-1,0} & w_{-1,1} \\ w_{0,-1} & w_{0,0} & w_{0,1} \\ w_{1,-1} & w_{1,0} & w_{1,1} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix},$$
   and the convolution is
$$\hat{x}_{i,j} = \sum_{k=-1}^{1}\sum_{l=-1}^{1} w_{k,l}x_{i-k,j-l}.$$
   (b) If we let w_v denote the vertical edge filter above, we can also define a horizontal edge filter
$$w_h = \frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let $\hat{v}$ and $\hat{h}$ denote the image x filtered with the vertical and horizonal filter, respectively. The Prewitt filtered image is obtained as

$$\hat{x}_{i,j} = \sqrt{\hat{v}_{i,j}^2 + \hat{h}_{i,j}^2}.$$

This filter is useful if we want to detect general edges in the image, and not only vertical edges.

(c) The image moment of order (p, q) is defined as

$$m_{p,q} = \sum_{i,j} i^p j^q x_{i,j}.$$

The corresponding central moment is

$$\mu_{p,q} = \sum_{i,j} (i - \bar{i})^p (j - \bar{j})^q x_{i,j},$$

where $\bar{i} = m_{1,0}/m_{0,0}$ and $\bar{j} = m_{0,1}/m_{0,0}$.

4. (a) The semivariogram is $\gamma(h) = r(0) - r(h) = 1 - e^{-h}$.

(b) To compute the kriging predictor, we first construct the covariance matrix of the observations,

$$\mathbf{\Sigma} = \begin{pmatrix} r(0) & r(\|s_1 - s_2\|) \\ r(\|s_2 - s_1\|) & r(0) \end{pmatrix} = \begin{pmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{pmatrix},$$

as well as the vector

$$\mathbf{c} = (r(\|s_0 - s_1\|), r(\|s_0 - s_2\|) = (r(1), r(\sqrt{2})) = (e^{-1}, e^{-\sqrt{2}}).$$

The kriging predictor is now given by

$$\begin{aligned}
\hat{X}(s_0) = \mathbf{c}\mathbf{\Sigma}^{-1}\mathbf{y} &= (e^{-1}, e^{-\sqrt{2}}) \begin{pmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \\
&= (e^{-1}, e^{-\sqrt{2}}) \frac{1}{1 - e^{-2}} \begin{pmatrix} 1 & -e^{-1} \\ -e^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \\
&= (e^{-1}, e^{-\sqrt{2}}) \frac{1}{1 - e^{-2}} \begin{pmatrix} 0.5 - e^{-1} \\ 1 - 0.5e^{-1} \end{pmatrix} \\
&= \frac{0.5e^{-1} - e^{-2} + e^{-\sqrt{2}} - 0.5e^{-(1+\sqrt{2})}}{1 - e^{-2}} \approx 0.2857.
\end{aligned}$$

(c) The variance is given by

$$\begin{aligned}
\sigma^2 &= r(0) - \mathbf{c}\mathbf{\Sigma}^{-1}\mathbf{c}^T \\
&= 1 - (e^{-1}, e^{-\sqrt{2}}) \frac{1}{1 - e^{-2}} \begin{pmatrix} 1 & -e^{-1} \\ -e^{-1} & 1 \end{pmatrix} \begin{pmatrix} e^{-1} \\ e^{-\sqrt{2}} \end{pmatrix} \\
&= 1 - (e^{-1}, e^{-\sqrt{2}}) \frac{1}{1 - e^{-2}} \begin{pmatrix} e^{-1} - e^{-(1+\sqrt{2})} \\ -e^{-2} + e^{-\sqrt{2}} \end{pmatrix} \\
&= 1 - \frac{e^{-2} - e^{-(2+\sqrt{2})} - e^{-(2+\sqrt{2})} + e^{-2\sqrt{2}}}{1 - e^{-2}} \\
&\approx 1 - 0.1488 = 0.8522.
\end{aligned}$$