Spatial statistics and image analysis, 7.5hp **Written exam**
TMS016/MSA301 **2019-06-05, 08:30 - 12:30**

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Allowed material: Chalmers allowed calculator.

Grading: Correct and well-motivated solutions (the first question only requires an answer) give the points written in parentheses at each question. You can in total get 20 points on the exam, and 20 points for the project assignments. To pass the course, you must have given a project seminar, have at least 8 points on the project assignments, and 8 points on the exam. Given that this is fulfilled, the limits for the grades are:
GU: 18 and 29 points for "G" and "VG" respectively.
CTH: 18, 26, and 32 points for 3,4, and 5 respectively.

Answers can be given in English or Swedish.

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1. Answer the following statements with one of the choices "true", "false", or "I do not know". For each statement, a correct answer gives 1 point, an incorrect answer (e.g. answering "true" when the statement is false) gives -0.5 points, and "I do now know" gives 0 points. (5p)

   (a) The spherical covariance function has compact support.

   (b) Assume that we have observations $Y_i = \mathbf{B}(\mathbf{s}_i)\boldsymbol{\beta} + X(\mathbf{s}_i), i = 1, \ldots, n$, of a geostatistical model with mean value parameters $\boldsymbol{\beta}$ and covariance parameters $\boldsymbol{\theta}$ for the Gaussian field $X(\mathbf{s})$. Two likelihood-based methods for estimation of the model parameters are profile likelihood estimation and REML estimation. REML gives an estimate of $\boldsymbol{\theta}$ with lower variance compared to the profile likelihood method.

   (c) The kriging predictor is unbiased.

   (d) The image opening with a circular structure element S is defined as $(A \oplus S) \ominus S$, where $\oplus$ denotes dilation and $\ominus$ denotes erosion.

   (e) Backpropagation is a method used to avoid overfitting in neural networks.

2. (a) Which property/properties does a function $r(\mathbf{s}, \mathbf{t})$, $\mathbf{s}, \mathbf{t} \in \mathbb{R}^2$, need to have to be a covariance function? (1p)

   (b) Assume that $X(\mathbf{s})$ is a Gaussian random field with covariance function $r(\mathbf{s}, \mathbf{t})$. If the field is stationary, which property does the covariance function possess? If the field also is isotropic, which further property does $r(\mathbf{s}, \mathbf{t})$ possess? (2p)

   (c) Show that we can add a nugget effect to the model and still have a valid covariance model. That is, if $r(\mathbf{s}, \mathbf{t})$ is a covariance function and

   $$r_\varepsilon(\mathbf{s}, \mathbf{t}) = \begin{cases} \sigma_e^2 & \text{if } \mathbf{s} = \mathbf{t}, \\ 0 & \text{if } \mathbf{s} \neq \mathbf{t}, \end{cases}$$

   then $r_Y(\mathbf{s}, \mathbf{t}) = r(\mathbf{s}, \mathbf{t}) + r_\varepsilon(\mathbf{s}, \mathbf{t})$ is also a covariance function. (2p)

3. Assume that we have a set of n images $\mathbf{x}_i$, $i = 1, \ldots, n$, from K different classes. For each image we know which class, $z_i \in \{1, \ldots, K\}$, it belongs to. We extract a vector $\mathbf{y}_i \in \mathbb{R}^p$ containing p different features that we want to use to train a classifier.

   (a) Write down and explain the model that assumed used when image classification is done using logistic regression. Also explain which function that is maximized when the parameters of the model are estimated from the data. (2p)

Figure 1: A noisy image.

(b) What is the difference between using LDA and logistic regression for classification? When is LDA preferable over logistic regression? (2p)

4. Assume that want to do noise reduction on the image in Figure 1.

(a) One option is to use image filtering. Explain how this works and propose a suitable filter. For your choice of filter, state the value of the filtered image in a certain pixel (i, j) as a function of the pixel values of the observed image. (2p)

(b) An alternative is to do the noise reduction using a Gaussian Markov random field model. Assume that we use a GMRF $\mathbf{X} \sim \mathsf{N}(\mu\mathbf{1}, \tau\mathbf{Q}^{-1})$ specified using a constant mean value $\mu\mathbf{1} = (\mu, \ldots, \mu)^T$ and the stencil

$$\mathbf{q} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

What is then the conditional distribution for the value of $\mathbf{X}$ at a given pixel (i, j) conditionally on the values at all other pixels? (2p)

(c) State a complete statistical model including $\mathbf{X}$ that can be used for noise reduction. (1p)

(d) Given that we chosen/estimated the parameters in the model, what is the conditional distribution, $\pi(\mathbf{X}|\mathbf{Y})$, for the field given the observed pixel values $\mathbf{Y}$? State the equation that is used to obtain the estimate of the noise free image. (1p)

Good luck!