TMA970 Inledande matematisk analys F1/TM1

Lösningar 27/10-2021

1. (a) divergent ; (b) konvergent ;
(c) konvergent ; (d) falskt ;
(e) falskt ; (f) falskt .

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2. (a) Sätt $t = \frac{\pi}{2} - x \iff x = \frac{\pi}{2} - t$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sin\left(\frac{\pi}{2} - t\right)}{\cos\left(\frac{\pi}{2} - t\right)} = \frac{\cos t}{\sin t}$$

$$\left(\frac{\pi}{2} - x\right)\tan x = t\,\frac{\cos t}{\sin t} = \frac{t}{\sin t}\cdot\cos t$$

$$x \to \frac{\pi}{2} \iff t \to 0$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2} - x\right)\tan x = \lim_{t\to 0}\underbrace{\frac{t}{\sin t}}_{\to 1}\cdot\underbrace{\cos t}_{\to 1} = 1$$

(b) $$\frac{\ln(\cos x)}{x^2} = \frac{\ln(1 + (\cos x - 1))}{x^2} = \text{"}\frac{0}{0}\text{"}$$

$$= \frac{\ln(1 + (\cos x - 1))}{\cos x - 1}\cdot\frac{\cos x - 1}{x^2} =$$

$$= -\frac{\ln(1 + (\cos x - 1))}{\cos x - 1}\cdot\frac{\overbrace{(1 - \cos x)(1 + \cos x)}^{= 1 - \cos^2 x = \sin^2 x}}{x^2\cdot(1 + \cos x)} =$$

$$= -\frac{\ln(1 + (\cos x - 1))}{\cos x - 1}\cdot\frac{\sin^2 x}{x^2}\cdot\frac{1}{1 + \cos x} \xrightarrow[x\to 0]{} -1\cdot 1\cdot\frac{1}{2},$$

3. $f(x) = e^{|\arctan x|}$ (2)

$D_f = \mathbb{R}$, $f > 0$ i $\mathbb{R}$

arctan $\cdot$ udda $\Rightarrow$ $|\arctan \cdot|$ jämn

$\Rightarrow$ $f(x)$ jämn

$\Rightarrow$ grafen symmetrisk m.a.p. y-axeln

$$\lim_{x \to \pm\infty} e^{|\arctan x|} = e^{|\pm\frac{\pi}{2}|} = e^{\frac{\pi}{2}}$$

$\Rightarrow$ sned (horisontell) asymptot i $\pm\infty$

inga vertikala asymptoter, eftersom f kontinuerlig i hela $\mathbb{R}$.

$x > 0$: $f(x) = e^{\arctan x}$,

$$f'(x) = e^{\arctan x} \cdot \frac{1}{1+x^2} > 0 \quad \forall x \in \mathbb{R} \text{ s.a. } x > 0$$

$\Rightarrow$ f växande (strängt) i $(0, \infty)$

$x < 0$: $f(x) = e^{-\arctan x}$

$$f'(x) = e^{-\arctan x} \cdot \left(-\frac{1}{1+x^2}\right) < 0 \quad \forall x \in \mathbb{R} \text{ s.a. } x < 0$$

$\Rightarrow$ f (strängt) avtagande i $(-\infty, 0)$

(man kan även se det per symmetri)

$\Rightarrow$ f har lok. min i $x_0 = 0$

? $\exists f'(0)$

$$\frac{f(h) - f(0)}{h} = \frac{e^{|\arctan h|} - 1}{h}$$

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$h>0$: $\dfrac{e^{\arctan h}-1}{h} =$

$= \dfrac{e^{\arctan h}-1}{\arctan h} \cdot \dfrac{\arctan h}{h} =$

$= \underbrace{\dfrac{e^{\arctan h}-1}{\arctan h}}_{\downarrow} \cdot \underbrace{\dfrac{\arctan h}{\sin(\arctan h)}}_{\downarrow} \cdot \underbrace{\cos(\arctan h)}_{\downarrow}$

1, ty $\arctan h \xrightarrow[h\to 0]{} 0$ $\quad 1 \quad 1$

$\xrightarrow[h\to 0_+]{} 1$

$h<0$: $\dfrac{e^{-\arctan h}-1}{h} = \dfrac{e^{-\arctan h}-1}{-\arctan h} \cdot \dfrac{-\arctan h}{h}$

$\xrightarrow[h\to 0_-]{} -1$

$\Rightarrow \nexists\, f'(0)$

$$f''(x) = \begin{cases} e^{\arctan x} \cdot \dfrac{1}{(1+x^2)^2} - e^{\arctan x} \cdot \dfrac{2x}{(1+x^2)^2}, & x>0 \\ e^{-\arctan x} \cdot \left(-\dfrac{1}{1+x^2}\right)^2 + e^{-\arctan x} \cdot \dfrac{2x}{(1+x^2)^2}, & x<0 \end{cases}$$

$x>0$: $f''(x) = \underbrace{e^{\arctan x}}_{>0} \cdot \underbrace{\dfrac{1}{(1+x^2)^2}}_{>0} (1-2x)$

$f''=0$: $x=\frac{1}{2}$

x	0		$\frac{1}{2}$	
f''		$+$	0	$-$

$\Rightarrow f$ konvex i $(0, \frac{1}{2})$, konkav i $(\frac{1}{2}, \infty)$

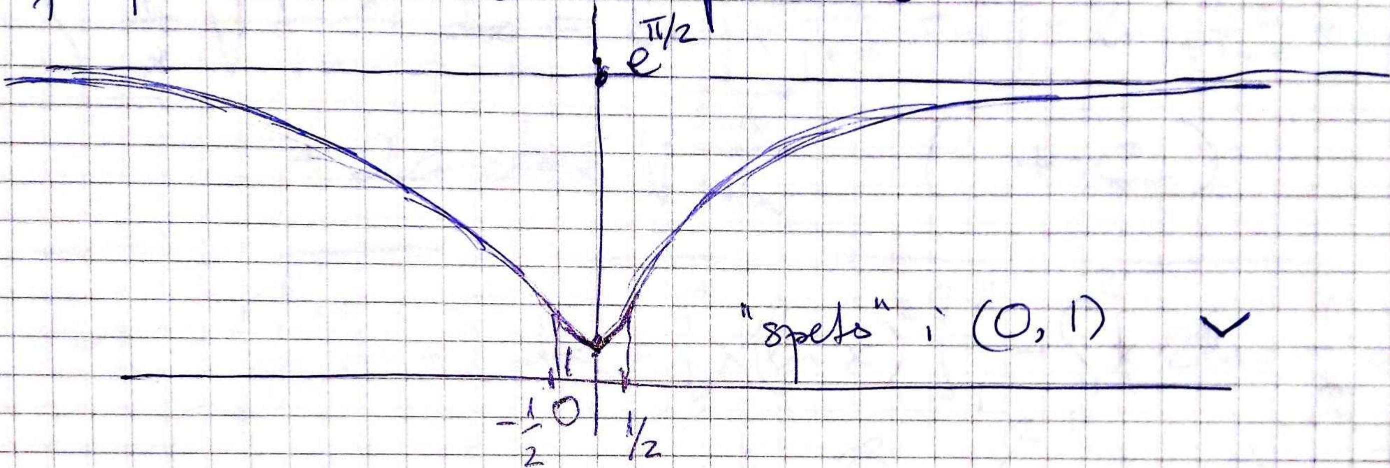
inflexion för $x=\frac{1}{2}$

$x<0$ (per symmetri)

f konkav i $(-\infty, -\frac{1}{2})$

f konvex i $(-\frac{1}{2}, 0)$, inflexion för $x=-\frac{1}{2}$

x	$-\infty$		$-\frac{1}{2}$		0		$\frac{1}{2}$		$+\infty$
f	$e^{\frac{\pi}{2}}$	$\searrow$	infl.		lok. min 1		infl.	$\nearrow$	$e^{\frac{\pi}{2}}$
f'		$-$	$-$		|	$+$	$+$		
f''		$-$	$-$ 0	$+$	|	$+$	0	$-$	$-$



"spets" i $(0,1)$ ✓

④ (a)
$$\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx = \left[\begin{array}{l|l} t=\sqrt{\frac{1-x}{1+x}} & x=\frac{1-t^2}{1+t^2} \\ t^2=\frac{1-x}{1+x} & dx=\frac{-4t}{(1+t^2)^2}\,dt \end{array}\right]$$

$$= \int \frac{\cancel{1+t^2}}{1-t^2}\cdot t \cdot \frac{-4t}{(1+t^2)^{\cancel{2}}}\,dt = +4\int \frac{t^2}{(-1+t^2)(1+t^2)}\,dt$$

$$\frac{t^2}{(1+t^2)(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{1+t^2}$$

$$t^2 = A(t+1)(1+t^2) + B(t-1)(1+t^2) + (Ct+D)(-1+t^2)$$

$t=1$: $\quad 1 = 4A \Rightarrow A = 1/4$

$t=-1$: $\quad 1 = -4B \Rightarrow B = -1/4$

t^3: $\quad 0 = A+B+C \Rightarrow C = 0$

t^0: $\quad 0 = A-B-D \Rightarrow D = \frac{1}{2}$

$$4\int \frac{t^2}{(t^2-1)(1+t^2)}\,dt = 4\cdot\frac{1}{4}\int\frac{dt}{t-1} -$$ 5

$$-4\cdot\frac{1}{4}\int\frac{dt}{t+1} + 4\cdot\frac{1}{2}\int\frac{dt}{1+t^2} =$$

$$= \ln|t-1| - \ln|t+1| + 2\arctan t \ (+C)$$

Vi sätter in $t = \sqrt{\frac{1-x}{1+x}}$:

$$\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx = \ln\left|\sqrt{\frac{1-x}{1+x}} - 1\right| - \ln\left|\sqrt{\frac{1-x}{1+x}} + 1\right| +$$

$$+ 2\arctan\sqrt{\frac{1-x}{1+x}} \quad (+C)$$

(b) $$\int_1^2 1\cdot(\ln x)^2\,dx = \left[x(\ln x)^2\right]_1^2 - \int_1^2 \cancel{x}\cdot 2\ln x\cdot\frac{1}{\cancel{x}}\,dx$$

$$= 2(\ln 2)^2 - 0 - \left(\left[2x\ln x\right]_1^2 - 2\int_1^2 \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right) =$$

$$= 2(\ln 2)^2 - 4\ln 2 + 0 + \left[2x\right]_1^2 =$$

$$= 2(\ln 2)^2 - 4\ln 2 + 4 + (-2) =$$

$$= 2(\ln 2)^2 - 4\ln 2 + 2$$

(c) $$\int_0^{\frac{\pi}{2}} \ln(\underbrace{\cos x}_{\substack{\searrow 0 \\ x\to\frac{\pi}{2}}})\,dx = \lim_{\varepsilon\to 0_+}\int_0^{\frac{\pi}{2}-\varepsilon} \ln(\cos x)\,dx =$$

$$x = \frac{\pi}{2} - t$$

$$= \lim_{\varepsilon\to 0_+}\int_{\frac{\pi}{2}}^{\varepsilon} \ln(\sin t)\cdot(-1)\,dt = \lim_{\varepsilon\to 0_+}\int_{\varepsilon}^{\frac{\pi}{2}} \ln(\sin t)\,dt =$$

$$= \lim_{\varepsilon\to 0_+}\int_{\varepsilon}^{\frac{\pi}{2}} \sqrt{\sin t}\,\ln(\sin t)\cdot\frac{\sqrt{t}}{\sqrt{\sin t}}\cdot\frac{1}{\sqrt{t}}\,dt$$

$$\sqrt{\sin t}\cdot\ln(\sin t)\xrightarrow[t\to 0]{} 0-$$

$$\sqrt{\frac{t}{\sin t}}\xrightarrow[t\to 0]{} 1$$

$$\Rightarrow \left|\sqrt{\sin t}\cdot\ln(\sin t)\cdot\sqrt{\frac{t}{\sin t}}\cdot\frac{1}{\sqrt{t}}\right| =$$

$$= \left|\sqrt{\sin t}\cdot\ln(\sin t)\right|\cdot\sqrt{\frac{t}{\sin t}}\cdot\frac{1}{\sqrt{t}} <$$

$$< \frac{1}{2}\cdot\frac{1}{\sqrt{t}} \quad \text{för } \underline{t} \text{ nära } 0$$

$$\int_0 \frac{dt}{\sqrt{t}} \quad \text{konvergent}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin t)\,dt \quad \text{konvergent}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln(\cos t)\,dt \quad \text{konvergent}$$

5.

$$\frac{\tan x-\sin x}{\left(\sqrt{1+x}-\sqrt{1-x}\right)^\alpha} = \frac{\sin x\,(1-\cos x)}{\cos x\left(\sqrt{1+x}-\sqrt{1-x}\right)^\alpha} =$$

$$= \frac{\sin x\,(1-\cos x)(1+\cos x)}{\cos x\,(1+\cos x)\left(\sqrt{1+x}-\sqrt{1-x}\right)^\alpha} = \frac{\sin^3 x}{\cos x\,(1+\cos x)\left(\sqrt{1+x}-\sqrt{1-x}\right)^\alpha}$$

$$= \frac{\sin^3 x\overbrace{\left(\sqrt{1+x}+\sqrt{1-x}\right)^\alpha}^{\to 2^\alpha}}{\underbrace{\cos x}_{\to 1}\,\underbrace{(1+\cos x)}_{\to 2}\,(2x)^\alpha} \longrightarrow \begin{cases} 0 & \text{för } \alpha<3 \\ 1/2 & \alpha=3 \\ \infty & \text{för } \alpha>3 \end{cases}$$

$$\Rightarrow L = \frac{1}{2}.$$

6. f kontinuerlig i $\mathbb{R}$

$\Rightarrow f$ har en primitiv i $\mathbb{R}$

$x \in [0, \pi]$: $|\sin x| = \sin x$

en primitiv : $-\cos x + C_1$

$x \in [-\pi, 0]$: $|\sin x| = -\sin x$

en primitiv : $\cos x + C_2$

C_1 och C_2 måste väljas så att den primitiva till $|\sin x|$ i $[-\pi, \pi]$ blir kontinuerlig i 0

$$\lim_{x \to 0_-} (\cos x + C_2) = 1 + C_2$$

$$\lim_{x \to 0_+} (-\cos x + C_1) = -1 + C_1$$

$$\Rightarrow 1 + C_2 = -1 + C_1$$

$\Rightarrow$ vi kan välja t.ex. $\begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$

I 0 :

$$\frac{(\pm\cos x \mp 1) - (1-1)}{x} \xrightarrow[x \to 0]{} 0$$

$\Rightarrow \exists$ derivata i 0 och den är $= 0 = \sin 0$

$\Rightarrow f(x) = \begin{cases} \cos x - 1 & x < 0 \\ -\cos x + 1 & x \geq 0 \end{cases}$ är en primitiv till $|\sin x|$ i $[-\pi, \pi]$

skarvas på samma sätt i alla pkter $k\pi$, $k \in \mathbb{Z}$