Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	22–01–04
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English-Swedish dictionary
Number of questions:	7; a passed question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Ann-Brith Strömberg, Emil Gustavsson (070 290 83 00)

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{lrl} \text{maximize} & z = x_1 + 2x_2, & \\ \text{subject to} & x_1 + x_2 & \geq -1, \\ & x_1 - x_2 & \geq 1, \\ & x_1, \quad x_2 & \geq 0. \end{array}$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) If an optimal solution exists, then use your calculations to decide whether it is unique or not. If the problem is unbounded, then use your calculations to specify a direction of unboundedness of the objective value.

(3p) Question 2

(Lagrangian duality and convexity)

Consider the problem to find

$$\begin{aligned} f^* = \text{infimum} \quad & (x_1 - 1)^2 - 2x_2, \\ \text{subject to} \quad & x_1 - 2x_2 \geq -2, \qquad \text{(C)} \\ & x_1, \ x_2 \geq 0. \end{aligned}$$

Lagrangian relax the constraint (C), and evaluate the dual function q at $\mu = 0$ and $\mu = 2$. Provide a bounded interval containing f^*.

(3p) **Question 3**

(modelling)

The *set covering problem* is a classical question in combinatorics, computer science and complexity theory. Given a set of elements $\mathcal{U} = \{1, 2, ..., n\}$ (called the universe) and a collection $\mathcal{S}$ of m sets whose union equals the universe, the *set cover problem* is the problem to identify the smallest sub-collection of $\mathcal{S}$ whose union equals the universe.

For example, consider the universe $\mathcal{U} = \{1, 2, 3, 4, 5\}$ and the collection of sets $\mathcal{S} = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$. Clearly the union of $\mathcal{S}$ is $\mathcal{U}$. However, we can cover all of the elements with the following, smaller number of sets: $\{\{1, 2, 3\}, \{4, 5\}\}$. This is also the smallest sub-collection whose union is $\mathcal{U}$.

A generalization of this problem is the *weighted set covering problem* where each set in $\mathcal{S}$ has a cost associated with it. The objective in the *weighted set covering problem* is to find a sub-collection of $\mathcal{S}$ whose union equals the universe, and so that the sum of the costs of the sets in the sub-collection is minimized.

Formulate an integer linear program (a linear objective function, linear constraints, and integrality restrictions on the variables) which models the weighted set covering problem.

Question 4

(True or False)

The below three claims should be assessed. For each claim: state whether it is true or false. Provide an answer together with a short but complete motivation.

(1p) a) Suppose the function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a vector $\boldsymbol{x} \in \mathbb{R}^n$.

Claim: for the vector $\boldsymbol{p} \in \mathbb{R}^n$ to be a descent direction with respect to f at $\boldsymbol{x}$ it is necessary that $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{p} < 0$.

(1p) b) *Claim:* For the phase I (when a BFS is *not* known a priori) problem of the simplex algorithm, the optimal value is always zero.

(1p) c) Consider a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$.

Claim: If f is differentiable at a point $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, then the identity $\partial f(\bar{\boldsymbol{x}}) = \{\nabla f(\bar{\boldsymbol{x}})\}$ holds.

Question 5

(unconstrained optimization)

Consider the unconstrained problem to minimize the function

$$f(x_1, x_2) = x_1^2 + x_1 x_2 - x_2^2 + 2x_1$$

(1p) a) Start in $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform two iterations with the steepest descent method using the step length $\alpha_k = 1$ in each iteration. Is the point reached an optimal solution?

(2p) b) Start in $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform two iterations with the Newton method using the Levenberg-Marquardt modification with $\gamma = 3$. Use step length $\alpha_k = 1$ in each iteration. Is the point reached an optimal solution?

Question 6

(Karush-Kuhn-Tucker)

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := -(x_1 - 3)^2 - (x_2 - 1)^2, \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(1p) a) State the KKT-conditions for the problem and verify that they are necessary.

(2p) b) Find all KKT-points, both graphically and analytically. What is the global optimum?

(3p) Question 7

(Farkas' lemma)

Farkas' Lemma can be states as follows:

Let $\boldsymbol{A}$ *be any* $m \times n$ *matrix and* $\boldsymbol{b}$ *an* $m \times 1$ *vector. Then exactly one of the two systems*

$$\begin{aligned} \boldsymbol{A}x &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \mathbf{0}^n, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \mathbf{0}^m, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned}$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.