Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	21–10–28
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English-Swedish dictionary
Number of questions:	7; a passed question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
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Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

(3p) Question 1

(Simplex method)

Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & z = x_1 - \ \ x_2, \\ \text{subject to} \quad & x_1 + \ \ x_2 \geq 1, \\ & x_1 + 2x_2 \leq 4, \\ & \qquad\ \ x_2 \geq 0. \end{aligned}$$

Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables used in the standard form.

Question 2

(LP duality)

Consider the linear programming problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{c} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m, \boldsymbol{l} \in \mathbb{R}^n$, and $\boldsymbol{u} \in \mathbb{R}^n$

(2p) a) Construct the LP dual of this problem.

(1p) b) Show that the dual problem is always feasible.

(3p) ## Question 3

(characterization of convexity in C^1)

Let $f \in C^1$ on an open convex set S. Establish the following characterization of the convexity of f on S:

$$f \text{ is convex on } S \quad \Longleftrightarrow \quad f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}), \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in S$$

Question 4

(True or False)

The below three claims should be assessed. For each claim: state whether it is true or false. Provide an answer together with a short but complete motivation.

(1p) a) Assume that you have solved a linear program (LP) with the simplex method, and at the found optimal solution all reduced costs of the non-basic variables are strictly positive.

Claim: The solution found is the unique optimal solution to the problem (i.e., there do not exist multiple optimal solutions).

(1p) b) Suppose $f \in C^2$. Assume that at some iteration point $\boldsymbol{x} \in \mathbb{R}^n$, there exists a solution $\boldsymbol{p}$ to the search direction-finding problem of Newton's method.

Claim: The direction $\boldsymbol{p}$ defines a descent direction to f at $\boldsymbol{x}$.

(1p) c) Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}$ where $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ is a symmetric and invertible matrix.

Claim: The function f is convex on $\mathbb{R}^n$.

Question 5

(exterior penalty method)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + x_2^2, \\ \text{subject to} \quad & h(\boldsymbol{x}) := x_1 + x_2 - 1 = 0. \end{aligned}$$

We consider solving the problem by using the exterior penalty method with the quadratic penalty function $\psi(s) = s^2$. The penalty problem is to

$$\underset{\boldsymbol{x} \in \mathbb{R}^n}{\text{minimize}}\, f(\boldsymbol{x}) + \nu \hat{\chi}_S(\boldsymbol{x}),$$

where $\hat{\chi}_S(\boldsymbol{x}) = \psi(h(\boldsymbol{x}))$, for positive, increasing values of the penalty parameter $\nu > 0$.

(1p) a) State the sequence of solutions to the penalty problem as a function of the penalty parameter ν.

(2p) b) Show that the sequence of solutions to the penalty problem converges to the unique optimal solution to the original problem when $\nu \to \infty$. Note: You need to show that the point the sequence converges to is optimal.

(3p) Question 6

(KKT)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}(x_1 - 5)^2 + \frac{1}{2}(x_2 - 3)^2, \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & 0 \leq x_j \leq 3, \quad j = 1, 2. \end{aligned}$$

Solve the problem using the KKT conditions (i.e., find the optimal solution to the problem and verify that it is optimal).

Are the KKT conditions necessary for optimality? Are the KKT conditions sufficient for optimality? Motivate your answers.

Question 7

(Modelling)

As a hedge fund manager you are responsible for choosing which stocks to include in the hedge funds portfolio. You have a budget of M amount to invest and you have n stocks to choose from ($\{1, \dots, n\}$). Each stock $i \in \{1, \dots, n\}$ has an expected return of $r_i > 0$ (i.e., if you invest x amount of money in stock i, the expected return of the investment is $r_i x$).

You must respect the following requirements:

- You may only invest in a maximum of k_{max} stocks.
- You must invest in a minimum of k_{min} stocks.
- You can not invest in both stock 1 and stock 2.
- The maximum investment you can make in any stock is $m \leq M$.

(2p) a) Formulate a *linear integer program* that maximizes the return of the portfolio.

(1p) b) Now you want to maximize the minimum return for any stock you invest in. This means that you would like to maximize the following entity instead:

$$\min_{i \in \{1,\dots,n\} \,:\, x_i > 0} r_i x_i,$$

where x_i if the amount you invest in stock i. Formulate this new problem as a *linear integer program*. Hint: You may need to introduce a variable z for the minimum return of the stocks you choose.