Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

Date: 21–10–28
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Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

Question 1

(Simplex method)

We first rewrite the problem on standard form. We rewrite $x_1 = x_1^+ - x_1^-$ and introduce slack variables s_1 and s_2.

$$\begin{aligned} \text{minimize} \quad & x_1^+ - x_1^- - x_2, \\ \text{subject to} \quad & x_1^+ - x_1^- + x_2 - s_1 = 1, \\ & x_1^+ - x_1^- + 2x_2 + s_2 = 4, \\ & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0. \end{aligned}$$

Phase I We introduce an artificial variable a and formulate our Phase I problem.

$$\begin{aligned} \text{minimize} \quad & a \\ \text{subject to} \quad & x_1^+ - x_1^- + x_2 - s_1 + a = 1, \\ & x_1^+ - x_1^- + 2x_2 + s_2 = 4, \\ & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0. \end{aligned}$$

We now have a starting basis (a, s_2). Calculating the reduced costs we obtain $\tilde{c}_N = (-1, 1, -1, 1)^{\mathrm{T}}$, meaning that x_1^+ or x_2 should enter the basis. We choose x_2. From the minimum ratio rest, we get that a should leave the basis. This concludes phase I and we now have the basis (x_2, s_2).

Phase II Calculating the reduced costs, we obtain $\tilde{c}_N = (2, -2, 1)^{\mathrm{T}}\mathrm{T}$, meaning that x_1^- should enter the basis. From the minimum ratio test, we get that the outgoing variable is s_2. Updating the basis we now have (x_1^-, x_2) in the basis.

Calculating the reduced costs, we obtain $\tilde{c}_N = (0, 3, 2)^{\mathrm{T}} \geq 0$, meaning that the current basis is optimal. The optimal solution is thus $(x_1^+, x_1^-, x_2, s_1, s_2) = (0, 2, 3, 0, 0)$ which in the original variables means $(x_1, x_2) = (-2, 3)$, with optimal objective value $f^* = -5$.

Question 2

(Duality)

The dual problem is the problem to

$$\begin{aligned}
\text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{l}^{\mathrm{T}}\boldsymbol{z}_l + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{z}_u, \\
\text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{z}_l + \boldsymbol{z}_u = \mathbf{c}, \\
& \boldsymbol{z}_l \geq \mathbf{0}, \\
& \boldsymbol{z}_u \leq \mathbf{0}.
\end{aligned}$$

This problem is always feasible since for any $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{l}, \boldsymbol{u}$, and $\boldsymbol{c}$ you can always set $\boldsymbol{y} = \mathbf{0}$ and solve the simple equation system $\boldsymbol{z}_l + \boldsymbol{z}_u = \boldsymbol{c}$ by letting $(\boldsymbol{z}_l)_i = c_i$ if $c_i > 0$, and $(\boldsymbol{z}_l)_i = 0$ otherwise. And letting $(\boldsymbol{z}_u)_i = c_i$ if $c_i < 0$, and $(\boldsymbol{z}_u)_i = 0$ otherwise.

Question 3

(characterization of convexity in C^1)

See Theorem 3.61 in textbook.

Question 4

(True or False)

a) True, since all reduced costs are positive no other variable can be included in the basis.

b) False, the direction may be an ascent direction.

c) False, the primal problem can be infeasible (if the dual is unbounded).

Question 5

(exterior penalty method)

(1p) a)

The penalty problem then becomes

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} \left(f(\boldsymbol{x}) + \nu h(\boldsymbol{x})^2\right) = \min_{\boldsymbol{x}\in\mathbb{R}^2} \left(x_1^2 + x_2^2 + \nu(x_1 + x_2 - 1)^2\right),$$

where $\nu > 0$. Noting that this is a convex function for positive ν we can solve the problem by setting the gradient to zero. Then we obtain

$$2x_1 + 2\nu x_1 + 2\nu x_2 - 2\nu = 0,$$
$$2x_2 + 2\nu x_1 + 2\nu x_2 - 2\nu = 0,$$

which has solution $\boldsymbol{x}_\nu = \frac{\nu}{1+2\nu}(1,1)^{\mathrm{T}}$.

(2p) b) Letting $\nu \to \infty$ we get that $\boldsymbol{x}_\nu \to (\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$. Analyzing the problem we see that this is a KKT point in the original problem. Since the original problem is convex, we have that the KKT conditions are sufficient for optimality, which means that the point $(\frac{1}{2}, \frac{1}{2})^{\mathrm{T}}$ is the optimal solution.

Question 6

(KKT)

Analyzing the problem we can see that at $\boldsymbol{x} = (3,2)^{\mathrm{T}}$ the constraints $g_1(\boldsymbol{x}) = x_1 + x_2 - 5 \leq 0$ and $g_2(\boldsymbol{x}) = x_1 - 3 \leq 0$ are active. We see that

$$\nabla f(\boldsymbol{x}) + \mu_1 \nabla g_1(\boldsymbol{x}) + \mu_2 \nabla g_2(\boldsymbol{x}) = \mathbf{0},$$

which at $\boldsymbol{x} = (3,2)^{\mathrm{T}}$ gives

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with solution $\mu_1 = \mu_2 = 1$.

Since all constraints are affine, we have that the affine constraints CQ is fulfilled, meaning that the KKT conditions are necessary for optimality for this problem.

Since the problem is convex, the KKT conditions are also sufficient for optimality. This implies that $\boldsymbol{x} = (3,2)^{\mathrm{T}}$ is the optimal solution to the problem.

Question 7

(Modelling)

(2p) a) Let x_i be the amount of money invested in stock i, $i \in \{1, \dots, n\}$. Let the amount invested in each stock be a nonzero integer number (i.e., $x_i \in \{0, 1, 2, \dots\}$) and introduce the binary variables $y_i \in \{0, 1\}$ denote if we invest any money in stock i or not. Then the problem can be formulated as

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in \{1,\dots,n\}} r_i x_i, \\
\text{subject to} \quad & x_i \leq m y_i, \quad i = 1, \dots, n, \\
& x_i \geq y_i, \quad i = 1, \dots, n, \\
& k_{\min} \leq \sum_{i \in \{1,\dots,n\}} y_i \leq k_{\max}, \\
& y_1 + y_2 \leq 1, \\
& x_i \in \{0, 1, 2, \dots\}, \quad i = 1, \dots, n, \\
& y_i \in \{0, 1\}, \quad i = 1, \dots, n.
\end{aligned}
$$

(1p) b) Introduce a variable z for the minimum return among all chosen stocks. Then the model needs to be altered by replacing the objective function with z and adding the constraints

$$z \leq r_i x_i + M(1 - y_i), \quad i = 1, \dots, n,$$

in order to represent z as the minimum return of the chosen stocks.