Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621 NONLINEAR OPTIMISATION

Date:	21–01-02
Time:	8^{30}–13^{30}
Aids:	All aids are allowed, but cooperation is not allowed
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Ann-Brith Strömberg

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

Question 1

(Simplex method)

Consider the problem to

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := -4x_1 + x_2, \\
\text{subject to} \quad & x_1 - x_2 \leq 2, \\
& -x_1 + 2x_2 \leq 1, \\
& x_1, \quad x_2 \geq 0.
\end{aligned}$$

(0.5p) a) Formulate the problem on the standard form for linear optimization problems.

(1.5p) b) Solve the problem using the simplex method. Present an optimal solution in the original variables.

(1p) c) Consider modifying the problem by including the variable x_3 as follows

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := -4x_1 + x_2 + x_3, \\
\text{subject to} \quad & x_1 - x_2 + x_3 \leq 2, \\
& -x_1 + 2x_2 - 3x_3 \leq 1, \\
& x_1, \quad x_2, \quad x_3 \geq 0.
\end{aligned}$$

Solve the problem using the simplex method using the optimal basis from b) as initial basis. Present an optimal solution or a ray of unboundedness in the original variables

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(3p)
Question 2

(Farkas Lemma)

Let $B, C \in \mathbb{R}^{m \times n}$ be matrices and $\boldsymbol{v} \in \mathbb{R}^m$ a vector. Assume that there exists a vector $\boldsymbol{z} \leq \boldsymbol{0}^n$ such that

$$B\boldsymbol{z} = C\boldsymbol{z} + \boldsymbol{v}.$$

Show that there cannot exist a vector $\boldsymbol{y} \in \mathbb{R}^m$ such $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{y} > 0$ and $C^{\mathrm{T}}\boldsymbol{y} \leq B^{\mathrm{T}}\boldsymbol{y}$.

Question 3

(KKT conditions)

Consider the following optimization problem, where $\boldsymbol{c}$ is a nonzero vector in $\mathbb{R}^n$:

$$\begin{aligned} &\max\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ &\text{s.t. } \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \leq 1. \end{aligned}$$

(1p) a) Show that $\bar{\boldsymbol{x}} = \boldsymbol{c}/||\boldsymbol{c}||$ is a KKT point.

(2p) b) Show that $\bar{\boldsymbol{x}}$ is the unique global optimal solution.

(3p) Question 4

(Gradient projection)

The gradient projection algorithm is a generalization of the steepest descent method to constrained optimization problems over convex sets. Given a feasible point $\boldsymbol{x}^k$, the next point is obtained according to $\boldsymbol{x}^{k+1} = \text{Proj}_X\left(\boldsymbol{x}^k - \alpha_k \nabla f(\boldsymbol{x})\right)$, where X is the convex set over which we minimize, $\alpha_k > 0$ is the step length, and $\text{Proj}_X(\boldsymbol{y}) = \arg\min_{\boldsymbol{x}\in X} ||\boldsymbol{x} - \boldsymbol{y}||$.

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) = x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 3x_2 + 8, \\ \text{subject to} \quad & \boldsymbol{x} \in X, \end{aligned}$$

where X is the rectangle $X = \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, 0 \leq x_1 \leq 3 \text{ and } 0 \leq x_2 \leq 2\}$

Start at the point $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step lengths $\alpha_k = 1$ for all k. You may solve the projection problem in the algorithm graphically. Is the point obtained a global/local minimum? Motivate why/why not.

(3p) Question 5

(Modelling)

Consider a Sudoku, i.e., a 3×3 matrix of cells where each cell is a 3×3 matrix of tiles; the Sudoku thus forms a 9×9 matrix of tiles. Each tile is to be assigned a number from one to nine such that the number is unique in the row, column, and cell containing the tile. The numbers of some tiles are given; an example of a Sudoku is illustrated in Figure 1.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Figure 1: A Sudoku.

(1.5p) a) Create a binary linear model of the feasible assignments of a Sudoku. Let x_{ijk} denote the binary decision choice of assigning number k to row i and column j, where $i, j, k \in \{1, \dots, 9\}$. Let $(i, j, k) \in A$ denote the set of initially given numbers, i.e., $x_{ijk} = 1$ for all $(i, j, k) \in A$.

hint: Introduce the sets C_l containing the tiles (i, j) in cell l, $l = 1, \dots, 9$.

(1.5p) b) Assume that the Sudoku has a feasible solution $\bar{\boldsymbol{x}}$. Add a linear objective function to your model in a) such that $\bar{\boldsymbol{x}}$ is an optimal solution if and only if it is the only feasible solution. Show that any other feasible solution $\tilde{\boldsymbol{x}} \neq \bar{\boldsymbol{x}}$ has a better objective value.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

(1p) a) Let S be a nonempty, closed and convex set in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be defined as $f(\boldsymbol{y}) = \min_{\boldsymbol{x} \in S} ||\boldsymbol{y} - \boldsymbol{x}||$.

Claim: The function f is convex.

(1p) b) *Claim:* If the KKT conditions are sufficient, then they are also necessary.

(1p) c) *Claim:* For the phase I (when a BFS is not known a priori) problem of the simplex algorithm, the optimal value is always zero.

(3p) Question 7

(Lagrangian relaxation and decomposition)

Consider the problem to

$$
\begin{aligned}
\text{minimize} \quad & z, && && (1)\\
\text{subject to} \quad & \sum_{j\in\mathcal{J}} p_{ij}x_{ij} \le z, && i\in\mathcal{I}, && (2)\\
& \sum_{i\in\mathcal{I}} x_{ij} = 1, && i\in\mathcal{J}, && (3)\\
& x_{ij}\in\{0,1\}, && j\in\mathcal{I}, j\in\mathcal{J}, && (4)\\
& z\in\mathbb{R}. && && (5)
\end{aligned}
$$

Here $\mathcal{I}$ denotes a set of machines and $\mathcal{J}$ denotes a set of tasks, x_{ij} denotes the decision to perform task j by machine i, and p_{ij} denotes the corresponding processing time. The variable z denotes the makespan, i.e., the time at which the last machine is finished.

(1p) a) Lagrangian relax constraints (2) with multipliers u_i, $i \in \mathcal{I}$. Let $h(\boldsymbol{u})$ denote the value of the dual function and show that $h(\boldsymbol{u}) = -\infty$ if $\sum_{i\in\mathcal{I}} u_i \neq 1$.

(1.5p) b) Assume that $\sum_{i\in\mathcal{I}} \bar{u}_i = 1$ and show that evaluating $h(\bar{\boldsymbol{u}})$ reduces to solving $\mathcal{J}$ separate optimization problems. State the optimal solution to each of these Lagrangian subproblems and the resulting formula for $h(\bar{\boldsymbol{u}})$.

(0.5p) c) Show that the Lagrangian subproblem solution forms a primal feasible solution for some value of z.