Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	20–10-29
Time:	8^{30}–13^{30}
Aids:	All aids are allowed, but cooperation is not allowed
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Ann-Brith Strömberg

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

Question 1

(Simplex method)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := |x_1| + |x_2|, \\ \text{subject to} \quad & x_1 - 2x_2 \geq 1, \\ & -x_1 - x_2 \leq 5. \end{aligned}$$

(1p) a) Rewrite the problem to standard form, by using the transformation $|x_i| = x_i^+ + x_i^-$ where $x_i = x_i^+ - x_i^-$ and $x_i^+, x_i^- \geq 0$, which is to be motivated in c). Then verify that using x_1^- and x_2^- as basic variables results in a basic feasible solution (BFS).

(1.5p) b) Solve the problem using the second phase of the simplex method. Use the BFS suggested in a) as the initial basis. Present the optimal solution in terms of the original variables.

(0.5p) c) Motivate the transformation made in a) by proving that the equality $|x_i| = x_i^+ + x_i^-$, $i = 1, 2$, holds in any BFS.

(3p) Question 2

(unconstrained optimization)

Consider the unconstrained problem to minimize the function

$$f(x_1, x_2) = x_1^2 + x_1 x_2 - x_2^2 + 2x_1$$

(1p) a) Start in $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform two iterations with the steepest descent method using the step length $\alpha_k = 1$ in each iteration. Is the point reached an optimal solution?

(2p) b) Start in $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform two iterations with the Newton method using the Levenberg-Marquardt modification with $\gamma = 3$. Use step length $\alpha_k = 1$ in each iteration. Is the point reached an optimal solution?

(3p) Question 3

Let $\phi : \mathbf{X} \times \mathbf{Y} \mapsto \mathbb{R}$ be a continuous function, where $\mathbf{X} \subseteq \mathbb{R}^n$ and $\mathbf{Y} \subseteq \mathbb{R}^m$ are non-empty sets.

(1p) a) Show that the following inequality holds

$$\sup_{\mathbf{y} \in \mathbf{Y}} \inf_{\mathbf{x} \in \mathbf{X}} \phi(\mathbf{x}, \mathbf{y}) \leq \inf_{\mathbf{x} \in \mathbf{X}} \sup_{\mathbf{y} \in \mathbf{Y}} \phi(\mathbf{x}, \mathbf{y}).$$

(2p) b) Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are nonempty, compact, and convex sets, and that the function ϕ is convex in $\mathbf{x}$ for any given $\mathbf{y}$ and concave in $\mathbf{y}$ for any given $\mathbf{x}$.

Show that the function $\rho : \mathbf{X} \mapsto \mathbb{R}$, defined by $\rho(\mathbf{x}) := \max_{\mathbf{y} \in \mathbf{Y}} \phi(\mathbf{x}, \mathbf{y})$, is a convex function in $\mathbf{x}$ and that the function $\delta : \mathbf{Y} \mapsto \mathbb{R}$, defined by $\delta(\mathbf{y}) := \min_{\mathbf{x} \in \mathbf{X}} \phi(\mathbf{x}, \mathbf{y})$, is a concave function in $\mathbf{y}$.

(3p) Question 4

(KKT conditions)

Consider the problem to

$$\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := -x_1 + x_2, \\
\text{subject to} \quad & x_1^2 + x_2 \leq 43, \\
& (x_1 - 1)^3 - x_2 \leq 0, \\
& x_1 \geq 2.
\end{aligned}$$

(2p) a) State the KKT-conditions for the problem and check whether they are necessary or not, and whether they are sufficient or not.

(1p) b) Find all KKT-points. For each of the KKT points, state whether it is optimal. Motivate!

(3p) Question 5

(Modelling)

Consider a network flow problem on a set of nodes N and a set of edges $E \subset N \times N$. Some nodes are source nodes $S \subset N$ in which a fluid enters the system. This fluid contains some pollutant. Let $\bar{p}_i$ denote the known portion of the pollutant in the fluid leaving node $i \in S$. Fluid can be purchased in the source nodes $i \in S$ at a cost of c_i SEK per unit of fluid. Then there are some intermediate nodes $I \subset N \setminus S$ that are pools in which the incoming fluids are mixed to a homogeneous state. Finally, there are some terminal nodes $T = N \setminus (I \cup S)$, which has a quantity demand $d_i \geq 0$ of fluid to be delivered and quality requirement represented by an upper bound $\bar{p}_i$ on the allowed portion of the pollutant. You can assume that the source nodes S can not have incoming flow from other nodes and that the terminal nodes T can not have outgoing flow to other nodes.

Construct a non-linear model minimizing the total purchase cost while satisfying the demand and the desired quality in all terminal nodes.

Hints: To model this you need two sets of variables: (i) let f_{ij} the denote the amount of fluid sent from a node $i \in N$ to a node $j \in N$ and (ii) let p_i denote the portion of the pollutant in the flow leaving node $i \in N$. The amount of the pollutant in the flow from a node i to a node j can thus be computed by the expression $f_{ij}p_i$.

All intermediate nodes have equal amounts of total incoming and outgoing fluid. And similarly, the total amount (not portion) of the pollutant entering and leaving these nodes must also be equal.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

(1p) a) Consider a primal-dual pair of linear programs.

Claim: If the dual problem is infeasible then the primal problem is unbounded.

(1p) b) Consider the problem to $\min_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x})$ where $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a twice differentiable function (i.e., $f \in C^2$).

Claim: If $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x}^*) \succeq 0$ then $\boldsymbol{x}^*$ is a local minimum of f.

(1p) c) Consider the Frank-Wolfe method used for minimizing a non-linear function over a polyhedron.

Claim: In each iteration of the algorithm a linear program needs to be solved in order to find the search direction.

(3p) Question 7

(Lagrangian duality)

Consider the problem (P) to

$$\begin{aligned} \text{minimize} \quad & x_1^2 + 2x_2^2, \\ \text{subject to} \quad & x_1 + x_2 \geq 2, \\ & x_1, x_2 \leq 2. \end{aligned}$$

Lagrangian relax the constraint $x_1 + x_2 \geq 2$. State and evaluate the Lagrangian dual function q at $\mu = 0$ and $\mu = 6$ and provide the corresponding lower bounds on the optimal objective value to the problem (P).
