Chalmers/GU
Mathematics sciences

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	20–08-18
Time:	8^{30}–13^{30}
Aids:	All aids are allowed, but cooperation is not allowed
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Ann-Brith Strömberg
Note:	It is not possible to "plus" (retaking an exam in a course you have already passed, to raise its grade). Students who have not yet passed the exam can attend this re-exam.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

Question 1

(Simplex method) Consider the problem

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \geq 2 \\ & x_1 - x_2 \leq 2 \\ & x_1, \quad x_2 \geq 0 \end{aligned}$$

(0.5p) a) Convert the problem to standard form.

(1.5p) b) Solve the problem using Phase I and Phase II of the simplex method. Use you calculations to provide an optimal solution or a unbounded ray in the original variables.

(1p) c) Derive the set of optimal solutions by analysing the reduced costs of the final iteration and conducting another minimum ratio test.

Question 2

(Representation theorem)

Consider the problem to $\text{minimize}_{\boldsymbol{x} \in P} f(\boldsymbol{x})$, where P is a non-empty polyhedron.

(2p) a) Assume that f is a concave function and that the problem has an optimal solution. Does the set of optimal solutions contain an extreme point of P? Prove or provide a counter example.

(1p) b) Assume that f is a convex function. Does the set of optimal solutions always contain an extreme point of P? Prove or provide a counter example.

(3p) Question 3

(Convexity)

Let $f_1, f_2, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Consider the function f defined by $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})\}$.

(2p) a) Show that f is convex.

(1p) b) State and prove a similar result for concave functions.

(3p) Question 4

(Linear programming)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned} \tag{1}$$

and the perturbed version of the problem where the right-hand-side is changed from $\boldsymbol{b}$ to $\boldsymbol{b} + \delta\boldsymbol{b}$. Show that if the original problem (1) has an optimal solution then the perturbed version cannot be unbounded, independently of $\delta\boldsymbol{b}$.

(3p) Question 5

(Modelling)

A rocket launching problem. Suppose that we are to send a rocket to the altitude of $\bar{z}$ [m] in time T [s]. Let $z(t)$ [m] denote its height above the ground at time t and $f(t)$ [N] be the non-negative upward force of the rocket thrusters at time t. Let the mass of the rocket be m [kg], the maximal thrust of the rocket be b [N], an let $v(t) = z'(t)$ [m/s] denote the upward velocity.

Formulate an optimization problem, with a quadratic objective function and affine constraints, that minimizes the energy required for the rocket to reach the desired altitude at time T.

Hints: The amount of energy required can be computed by

$$\int_0^T f(t)v(t)\,dt,$$

and the equation of motion is

$$mv'(t) + mg = f(t), \quad t \in [0, T].$$

Assume that the time interval is divided into K periods of length $l := T/K$ and let $f_k := f(lk)$, $z_k := z(lk)$, $v_k := v(lk)$, $k = 1, \ldots K$. Then approximate the velocity and acceleration using finite differences, e.g., $v_k = (z_k - z_{k-1})/l$, $k = 1, \ldots, K$. Similarly, approximate the integral as a Riemann sum.

Question 6

(true or false)

Indicate for each of the following three statements whether it is true or false. Motivate your answers!

(1p) a) *Claim:* The Simplex method is a suitable solution method for problems where a convex objective function should be optimized over a polytope.

(1p) b) *Claim:* For a convex optimization problem, every KKT-point is a global optimal solution.

(1p) c) Consider a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$.

Claim: If f is differentiable at a point $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, then the identity $\partial f(\bar{\boldsymbol{x}}) = \{\nabla f(\bar{\boldsymbol{x}})\}$ holds.

(3p) Question 7

(Exterior penalty method)

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := 2e^{x_1} + 3x_1^2 + 2x_1x_2 + 4x_2^2, \\ \text{subject to} \quad & 3x_1 + 2x_2 - 6 = 0. \end{aligned}$$

Formulate a suitable exterior penalty function with the penalty parameter $\nu = 10$. Starting at the point $(1, 1)$, perform one iteration of a gradient method to solve the unconstrained penalty problem.
