Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

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Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

Question 1

(the simplex method)

(1p) a) The dual problem in standard form becomes:

$$\begin{array}{lrrrrrrl}
\text{minimize} & z = 2y_1 + & y_2 + & \frac{1}{2}y_3 + & \frac{1}{2}y_4, & & & \\
\text{subject to} & 2y_1 & & -y_3 & +y_4 & -s_1 & & = 1, \\
 & y_1 + & y_2 & +y_3 & -y_4 & & -s_2 & = 1, \\
 & y_1, & y_2, & y_3, & y_4, & s_1, & s_2 & \geq 0.
\end{array}$$

(1.5p) b) Introducing the artificial variable a_1, phase I gives the problem

$$\begin{array}{lrrrrrrrl}
\text{minimize} & w = & & & & & & a_1, & \\
\text{subject to} & 2y_1 & & -y_3 & +y_4 & -s_1 & & +a_1 & = 1, \\
 & y_1 & +y_2 & +y_3 & -y_4 & & -s_2 & & = 1, \\
 & y_1, & y_2, & y_3, & y_4, & s_1, & s_2, & a_2 & \geq 0.
\end{array}$$

Using the starting basis $(a_1, y_2)^T$ gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} 2 & -1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced costs, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$, for this basis is $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} -2, & 1, & -1, & 1 & 0 \end{pmatrix}$, which means that y_1 enters the basis. $\boldsymbol{B}^{-1}\boldsymbol{N}_1 = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ thus the minimum ratio test implies that a_1 leaves.

Thus, we move on to phase II using the basis $\begin{pmatrix} y_1, & y_2 \end{pmatrix}^T$, and

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.$$

The new reduced costs are $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} 0, & 1, & \frac{1}{2}, & 1 \end{pmatrix}$. Since the reduced costs are all non-negative, the current BFS is optimal. The optimal solution to the dual problem is hence $\begin{pmatrix} y_1, & y_2, & y_3, & y_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}, & \frac{1}{2}, & 0, & 0 \end{pmatrix}$ with the objective value of $\frac{3}{2}$.

(.5p) c) Since the primal variables of our original problem are the dual variables of the dual problem, we get that $\boldsymbol{x}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = \begin{pmatrix} \frac{1}{2}, & 1 \end{pmatrix}$.

Question 2

(unconstrained optimization)

a) For the steepest descent method:

$$\boldsymbol{p} = -\nabla f(\boldsymbol{x}^0) = (-4, 0)^T$$

b) For Netwon's method:

$$\boldsymbol{p} = -[\nabla^2 f(\boldsymbol{x})]^{-1}\nabla f(\boldsymbol{x}^0) = (-4/3, -2/3)^T$$

c) For Levemberg-Marquardt method:

$$\boldsymbol{p} = -[\nabla^2 f(\boldsymbol{x}) + \gamma I]^{-1}\nabla f(\boldsymbol{x}^0) = (-4/9, 2/9)^T$$

The methods a) and c) always finds descent directions (if γ is chosen large enough)

(3p) Question 3

(Lagrangian relaxation)

Lagrangian relax the first constraint, we can get:

$$L(\boldsymbol{x},\ \mu) = x_1 - 2x_2 + \mu(2 - x_1 + x_2) = (1-\mu)x_1 + (\mu - 2)x_2 + 2\mu.$$

$$q(\mu) = \min_{\boldsymbol{x}} L(\boldsymbol{x},\ \mu) = \begin{cases} 7\mu - 10, & \mu \in [0, 1.5) & x_1 = 0, x_2 = 5, \\ 0.5, & \mu = 1.5 & x_1 + x_2 = 5, \\ 5 - 3\mu & \mu \in (1.5, \infty) & x_1 = 5, x_2 = 0. \end{cases}$$

So $q^* = 0.5$, $\mu^* = 1.5$. Since the constraints are affine, by strong duality, we can get $z^* = q^* = 0.5$.
For complementary slackness, we need to fulfill $\mu_i^* g_i(\boldsymbol{x}^*) = 0$, since $\mu \neq 0$, so $g_i(\boldsymbol{x}^*) = 0$, which means $2 - x_1 + x_2 = 0$. Combine with $x_1 + x_2 = 5$, we can get $x^* = (3.5, 1.5)^T$. We can check that (x^*, μ^*) fulfilled all the conditions listed in Theorem 6.8, so x^* is the optimal solution for the original problem. The optimal value is 0.5.

(3p) Question 4

(KKT conditions)

(2p) a) The KKT conditions are

$$\nabla f(\boldsymbol{x}) + \lambda \nabla h(\boldsymbol{x}) = \begin{pmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

There is only one feasible point fulfilling the KKT conditions:

$$\bar{\boldsymbol{x}} = (4, 4, 4)^T$$

with $\gamma = -8$.

(1p) b) The problem is undounded. Take $x_1 = M$, $x_2 = M$ and $x_3 = 12 - 2M$ which is feasible. The objective value is $x_1x_2 + x_1x_3 + x_2x_3 = M^2 + M(12 - 2M) + M(12 - 2M) = 24M - 3M^2$. Let M tend to infinity and you get an undounded solution.

(3p) Question 5

(modelling)

Variables, let

- x_{ij} equal to one if the piece of length l_i is cut from the board of length L_j, and equal to zero otherwise, $i = 1, \dots, N$, $j = 1, \dots, M$.
- y_j equal to one if the board of length L_j is purchased, $j = 1, \dots, M$.
- z_k be the number of times a discount has been retrieved for board of type k, $k = 1, \dots, K$.

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{M} p_j y_j - \sum_{k=1}^{K} d_k z_k, && && (1)\\
\text{s.t.} \quad & \sum_{i=1}^{N} l_i x_{ij} \leq L_j y_j, && j = 1, \dots, M && (2)\\
& \sum_{j=1}^{M} x_{ij} = 1, && i = 1, \dots, N, && (3)\\
& \sum_{j \in S_k} y_j \geq 4 z_k, && k = 1, \dots, K, && (4)\\
& x_{ij} \in \{0, 1\}, && i = 1, \dots, N,\ j = 1, \dots, M && (5)\\
& y_j \in \{0, 1\}, && j = 1, \dots, M. && (6)\\
& z_k \in \mathbb{Z}^+, && j = 1, \dots, K. && (7)
\end{aligned}$$

Question 6

(true or false)

(1p) a) *True.* The KKT conditions becomes

$$\nabla f(\boldsymbol{x}) + \sum_{i=1}^{3} \mu_i \nabla g_i(\boldsymbol{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} -x_2 \\ 2x_2 - x_1 + 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$g_i(\boldsymbol{x}) \leq 0,\ \mu_i \geq 0,\ \mu_i g_i(\boldsymbol{x}) = 0,\ i = 1, 2, 3$$

Where $\mu_2 > 0 \Rightarrow \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mu_2 = 0$ leads to an inconsistent system.

(1p) b) *True.* We check if the gradient cone and tangent cone are equal. The gradient cone is $G(\boldsymbol{x}^*) = \{\boldsymbol{p} \in \mathbb{R}^2 \,|\, x_2 \leq 0, x_1 \geq 0, x_2 \geq 0\} = \{\boldsymbol{p} \in \mathrm{R}^2 \,|\, x_1 \geq 0, x_2 = 0\}$. For the tangent cone, let $\{\boldsymbol{x}^k\} \subset S$ be any sequence of points converging to $\boldsymbol{x}^*$, thus for any $\varepsilon > 0$ $\exists K$ such that $\boldsymbol{x}_1^k \leq \varepsilon$, $\forall k \geq K$. Assuming that $\boldsymbol{x}_2^k > 0$ leads to a contradiction that $\boldsymbol{x}_1^k > 1$ thus $\boldsymbol{x}_2^k = 0$, $\forall k \geq K$. We thus get that $G(\boldsymbol{x}^*) = T_S(\boldsymbol{x}^*)$, i.e., Abadie's CQ holds.

(1p) c) *False.* Since any sequence of converging points must satisfy $\boldsymbol{x}_2^k = 0$, we have that there exist no sequence of strict interior points that converge to $\boldsymbol{x}^*$.

(3p) Question 7

(convergence of an exterior penalty method)

See Theorem 13.3 in the course book.
