Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MMG621
NONLINEAR OPTIMISATION

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Examiner: Michael Patriksson

Note: the solutions presented here are brief in relation to the requirements on your answers, in particular regarding your motivations.

(3p) **Question 1**

(the simplex method)

Rewrite the problem into standard form by and adding/subtracting slack variables s_1 and s_2 to the left-hand side in the first and second constraint, respectively. Moreover, let $z := -z$ to get the problem on minimization form. Thus, we get the following linear program:

$$\begin{array}{lrrrrl} \text{minimize} & z = -3x_1 & - \ 2x_2, & & & \\ \text{subject to} & 2x_1 & + \ 3x_2 & + s_1 & & = 1, \\ & x_1 & - \ x_2 & & - s_2 & = 4, \\ & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

Introducing the artificial variable a, phase I gives the problem

$$\begin{array}{lrrrrrl} \text{minimize} & w = a, & & & & & \\ \text{subject to} & 2x_1 & + 3x_2 & + s_1 & & & = 1, \\ & x_1 & - \ x_2 & & - s_2 & + a & = 4, \\ & x_1, & x_2, & s_1, & s_2, & a & \geq 0. \end{array}$$

Using the starting basis $(s_1, a)^T$ gives

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{N} = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \boldsymbol{x}_B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \boldsymbol{c}_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The reduced costs, $\bar{\boldsymbol{c}}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}$, for this basis is $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$, which means that x_1 enters the basis. The minimum ratio test implies that s_1 leaves.

Updating the basis we now have $(x_1, a)^T$ in the basis. Calculating the reduced costs, we obtain $\bar{\boldsymbol{c}}_N^T = \begin{pmatrix} 5/2 & 1/2 & 1 \end{pmatrix}$, meaning that the current basis is optimal. The optimal solution is thus $a^* = 7/2$, since $a^* > 0$, so the original problem is infeasble.

Since the original problem is infeasible, so it is neither existing an optimal solution nor unbounded.

Question 2

(1p) a) The LP dual problem is to:

$$\begin{aligned} \text{miximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ \text{subject to} \quad & A^{\mathrm{T}}y \leq c, \\ & y \quad \geq 0^m. \end{aligned}$$

(2p) b) If the dual problem has a finite optimal solution, then so does the primal problem. If the dual problem is unbounded, then the primal problem is infeasible. See Theorem 10.6 (Strong Duality Theorem).

Question 3

(feasible direction methods)

(2p) a) For the Frank-Wolfe algorithm, $y_1 = (1,0)^{\mathrm{T}}$, $x_1 = (0,1)^{\mathrm{T}}$, $y_2 = (0,0)^{\mathrm{T}}$, $x_2 = (9/20, 3/20)^{\mathrm{T}}$.

(1p) b) For the simplicial decomposition algorithm, $P_0 = \emptyset$, $y_1 = (1,0)^{\mathrm{T}}$, $P_1 = (1,0)^{\mathrm{T}}$, $x_1 = (3/4, 1/4)^{\mathrm{T}}$,$y_2 = (0,0)^{\mathrm{T}}$, $P_2 = (1,0)^{\mathrm{T}} \bigcup (0,0)^{\mathrm{T}}$, $x_2 = (1/2, 0)^{\mathrm{T}}$,

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(3p) **Question 4**

(on the SQP algorithm and the KKT conditions)

The result is based on a comparison between the KKT conditions of the original problem,

$$\begin{aligned} \text{minimize } & f(\boldsymbol{x}), && && \text{(1a)}\\ \text{subject to } & g_i(\boldsymbol{x}) \le 0, && i = 1, \dots, m, && \text{(1b)}\\ & h_j(\boldsymbol{x}) = 0, && j = 1, \dots, \ell, && \text{(1c)} \end{aligned}$$

and those of the SQP subproblem,

$$\begin{aligned} \underset{p}{\text{minimize }} & \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{B}_k\boldsymbol{p} + \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)}\\ \text{subject to } & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \le 0, && i = 1, \dots, m, && \text{(2b)}\\ & h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1, \dots, \ell. && \text{(2c)} \end{aligned}$$

We first note that the latter problem is a convex one (the matrix $\boldsymbol{B}_k$ was assumed to be positive semidefinite), and that the solution $\boldsymbol{p}_k$ is characterized by its KKT conditions, since the constraints are linear (so that Abadie's CQ is fulfilled). It remains to compare the two problems' KKT conditions. With $\boldsymbol{p}_k = \boldsymbol{0}^n$ they are in fact identical!

(3p) **Question 5**
(modelling)
Sets:
$I := \{1, ..., 10\}$, the set of schools.

The decision variables are:

$$x_i = \begin{cases} 1 & \text{keep school } i, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in I$.

$$y_{ij} = \begin{cases} 1 & \text{home area } j \text{ go to school } i, \\ 0 & \text{otherwise,} \end{cases}$$

where $i \in I$, $j \in J$.
Model:

$$\begin{aligned} \text{minimize} \quad & x_i c_i + m b_j d_{ij}, \\ \text{subject to} \quad & \sum_{i \in I} x_i \leq 9, \\ & \sum_{i \in I} x_i \geq 7, \\ & \sum_{j \in J} b_j y_{i,j} \leq k_i, && i \in I, \\ & y_{i,j} \leq x_i, && i \in I, j \in J, \\ & \sum_{i \in I} y_{i,j} = 1, && j \in J, \\ & x_i \in \{0, 1\}, && i \in I, \\ & d_{ij} \in \{0, 1\}, && i \in I, j \in J. \end{aligned}$$

Question 6
(true or false)

(1p) a) False. The original problem can be infeasible, which means the optimal value for phase I is higher than 0, like question 1 in this exam.

(1p) b) True. Since $\nabla f(\boldsymbol{x}) \neq \mathbf{0}^n$, and $\boldsymbol{G}$ is a symmetric and positive definite matrix of dimension $n \times n$, we have that $\nabla f(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{d} = -\nabla f(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{G}^{-1} \nabla f(\boldsymbol{x}) < 0$, so $\boldsymbol{d}$ is a decent direction. By defination of decent direction, the clam is true.

(1p) c) False. For example, $g(\boldsymbol{x}) = -x^2$ is concave, but $\{ \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) \leq -1 \}$ is not convex.

(3p)

Question 7

(the gradient projection algorithm)

$\nabla f(\boldsymbol{x}) = (2x_1 - 2x_2 - 2, 4x_2 - 2x_1 - 3)^{\mathrm{T}}$, $x_0 - \alpha_k \nabla f(\boldsymbol{x}_0) = (2, 3)^{\mathrm{T}}$, $x_1 = (2, 2)^{\mathrm{T}}$, $x_1 - \alpha_k \nabla f(\boldsymbol{x}_1) = (4, 1)^{\mathrm{T}}$, $x_2 = (3, 1)^{\mathrm{T}}$.

Since the feasible set is convex, there exists an interior point, so the Slater CQ holds. Since it is a convex problem, so the KKT conditions are both necessary and sufficient. $(3, 1)^{\mathrm{T}}$ is not a KKT point, so it is neither a global nor a local minimum.