Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
NONLINEAR OPTIMISATION

Date:	18–01–09
Time:	8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Michael Patriksson, tel. 709-581812
Result announced:	18–01–30
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

The following linear optimization problem is given:

$$\begin{aligned} \text{maximize} \quad & z = -x_1 - 2x_2, \\ \text{subject to} \quad & -x_1 + x_2 \leq 5, \\ & x_2 \geq 2. \end{aligned}$$

(1p) a) Rewrite the problem to standard form by adding slack variables to both constraints.

(2p) b) Solve the problem using phase I and phase II of the simplex method.

Aid: You may utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Question 2

(Lagrangian duality and convexity)

Consider the problem to find

$$\begin{aligned} f^* = \text{infimum} \quad & (x_1 - 1)^2 - 2x_2, \\ \text{subject to} \quad & x_1 - 2x_2 \geq -2, \qquad \text{(C)} \\ & x_1, x_2 \geq 0. \end{aligned}$$

(2p) a) Lagrangian relax the constraint (C), and evaluate the dual function q at $\mu = 0$ and $\mu = 2$. Provide a bounded interval containing f^*.

(1p) b) Show that for a general convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and any $\boldsymbol{x} \in \mathbb{R}^n$, the subdifferential $\partial f(\boldsymbol{x})$ is a convex set.

Question 3

(Karush-Kuhn-Tucker)

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := -(x_1 - 3)^2 - (x_2 - 1)^2, \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(1p) a) State the KKT-conditions for the problem and verify that they are necessary.

(2p) b) Find all KKT-points, both graphically and analytically. What is the global optimum?

(3p) Question 4

(unconstrained optimization)

Let $f(\boldsymbol{x}) := x_1^2 + 2x_1x_2 - 2x_2^2 + 4x_1$ and $\bar{\boldsymbol{x}} = (0, 0)^{\mathrm{T}}$. Find the search directions at $\bar{\boldsymbol{x}}$ for the following three unconstrained optimization methods:

a) Steepest descent method,

b) Newton's method,

c) Newton's method with the Levenberg-Marquardt modification using $\gamma = 8$ (where γ is the amount added to the diagonal of the Hessian).

In general, for which of the methods a)-c) are the directions found always *descent directions*? Motivate your answer.

(3p) Question 5

(modelling)

There are 7 wind turbines, which all need to be maintained once during the week. There are two maintenance teams: maintenance team 1 and maintenance team 2. There is no difference between the two maintenance teams. The maintenance teams only work on workdays, i.e. from Monday to Friday. It takes one maintenance team a full day to maintain one wind turbine. Due to different locations of each wind turbine and the weather of the date, the maintenance costs are different. The costs are stated in Table 1. The costs are the same for both maintenance teams.
Formulate an integer linear model to minimize the maintenance cost.

turbine	Mon	Tue	Wed	Thu	Fri
1	10	11	12	13	14
2	12	14	16	18	20
3	17	18	17	18	17
4	20	19	18	17	16
5	22	22	22	22	33
6	24	23	22	23	23
7	9	6	8	7	9

Table 1: Maintenance costs [10^3 \$] of different wind turbines in different days

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Question 6

(true or false)

The below three individual claims should be assessed individually. Are they true or false, or is it impossible to say? For each of the three statements, provide an answer, together with a short—but complete— motivation.

(1p) a) Consider a minimization problem, where the objective function is convex, and the feasible set is

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid g_i(\boldsymbol{x}) \leq 0, \ i = 1, \dots, m; \ \ h_i(\boldsymbol{x}) = 0, \ i = 1, \dots, k\}, \qquad (1)$$

where $g_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \dots, m$, and $h_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \dots, k$ are convex functions.

Claim: The problem is a convex optimization problem.

(1p) b) Let $f(\boldsymbol{x}) := \ln \left(\sum_{j=1}^{n} e^{a_j x_j} \right)$, where $a_j \in \mathbb{R}, j = 1, ..., n$ are constants;

Claim: f is a convex function.

(1p) c) Consider the program

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, m, \end{aligned}$$

where the functions f and g_i, $i = 1, \ldots, m$, are convex. Suppose that $\boldsymbol{x}^*$ is a globally optimal solution to this problem, and that $g_k(\boldsymbol{x}^*) < 0$ for some index $k \in \{1, \ldots, m\}$.

Claim: If we remove constraint k from the problem its set of optimal solutions is unchanged.

(3p) Question 7

(LP duality)

Consider the following standard form of a linear program:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{c} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.
