Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	16–01–12
Time:	Eklandagatan 86, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Edvin Wedin, tel. 0703-088304
Result announced:	16–02–02
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

(3p) Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{llrcl} \text{minimize} & z = & 2x_1 - x_2 + x_3, & & \\ \text{subject to} & & x_1 + 3x_2 - x_3 & \leq & 5, \\ & & -2x_1 + x_2 - 2x_3 & \leq & -2, \\ & & x_1, \quad x_2, \quad x_3 & \geq & 0. \end{array}$$

(2p) a) Solve the problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables in the standard form.
Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Suppose that to the original problem we add a new variable x_4 and obtain the new problem to

$$\begin{array}{llrcl} \text{minimize} & z = & 2x_1 - x_2 + x_3 - \frac{1}{2}x_4, & & \\ \text{subject to} & & x_1 + 3x_2 - x_3 + 8x_4 & \leq & 5, \\ & & -2x_1 + x_2 - 2x_3 - x_4 & \leq & -2, \\ & & x_1, \quad x_2, \quad x_3, \quad x_4 & \geq & 0. \end{array}$$

If the original problem has an optimal solution, explain how the optimal solution is affected by adding the new variable. If the original problem is unbounded, investigate if adding the new variable affects the unboundedness of the problem.

Note: Use your calculations from a) to answer the question.

(3p) **Question 2**

(Quadratic programming)

Consider the minimization of the quadratic function $f(\boldsymbol{y}) := \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} + c$ subject to the constraints $\boldsymbol{y} \geq \boldsymbol{0}^n$, where $\boldsymbol{A}$ is symmetric and positive semi-definite. Show that the three conditions $\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b} \geq \boldsymbol{0}^n$, $\boldsymbol{x} \geq \boldsymbol{0}^n$, and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{0}$ are necessary and sufficient to characterize $\boldsymbol{x}$ as a minimum.

(3p) **Question 3**

(characterization of convexity in C^1)

Let $f \in C^1$ on an open convex set S. Establish the following characterization of the convexity of f on S:

f is convex on $S \Longleftrightarrow f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x})$, for all $\boldsymbol{x}, \boldsymbol{y} \in S$.

Question 4

(true or false claims in optimization)

For each of the following three claims, your task is to decide whether it is true or false. Motivate your answers clearly.

(1p) a) If, in the solution of a minimization problem of the linear programming type, a current non-degenerate BFS (basic feasible solution) has a non-negative vector of reduced costs, then that BFS corresponds to an optimal extreme point solution in the problem.

(1p) b) If, in the solution of an unconstrained optimization problem of the form

$$\underset{x \in \mathbb{R}^n}{\text{minimize}}\, f(\boldsymbol{x}),$$

with $f : \mathbb{R}^n \to \mathbb{R}$ being in C^1, you have found a vector $\bar{\boldsymbol{x}}$ with $\nabla f(\bar{\boldsymbol{x}}) = \boldsymbol{0}^n$, then $\bar{\boldsymbol{x}}$ is an optimal soution to the problem.

(1p) c) Suppose you have solved the problem to

$$\underset{x \in X}{\text{minimize}}\, f(\boldsymbol{x}),$$

where $X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, m \,\}$, the functions f and g_i, $i = 1, \dots, m$, are continuous, and the vector $\bar{\boldsymbol{x}}$ is an optimal solution. Suppose further that for some $j = 1, \dots, m$, $g_j(\bar{\boldsymbol{x}}) < 0$. Then, that constraint is *redundant*, that is, if that constraint is removed, then the remaining problem with $m-1$ constraints also has $\bar{\boldsymbol{x}}$ as an optimal solution.

(3p) **Question 5**

(KKT conditions)

Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 2 \\ & (x_1 - 2)^2 + (x_2 - 2)^2 \leq 2 \end{aligned}$$

(1p) Establish theoretically or graphically that $\boldsymbol{x}^* = (1, 1)^{\mathrm{T}}$ is the unique globally optimal solution.

(2p) Determine if the KKT conditions are satisfied at $\boldsymbol{x}^*$. If they are not, explain why, and relate your explanation to the known results on necessary and sufficient optimality conditions.

(3p) **Question 6**

(Frank-Wolfe algorithm)

Consider the problem

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & f(\boldsymbol{x}) := \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 52 & 34 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{subject to} \quad & x_1 + 2x_2 \leq 4 \\ & x_1 + x_2 \leq 3 \\ & 2x_1 \leq 5 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned} \tag{1}$$

Solve problem (1) with the Frank-Wolfe algorithm. Start with initial guess $\boldsymbol{x}^{(0)} = (x_1, x_2)^{\mathrm{T}} = (0,0)^{\mathrm{T}}$. Use exact minimization for line search. If necessary, you are allowed to carry out the calculations approximately with two digits of accuracy.
Hint: You may find it helpful to analyze the problem and the algorithm progress in picture, but this should be augmented with rigorous analysis.

(3p) Question 7

(LP duality)

Consider the problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{maximize}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \inf_{\boldsymbol{y} \in P} \ \boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} \geq d \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where the problem data are $\boldsymbol{c} \in \mathbb{R}^n$, $d \in \mathbb{R}$, $P = \{\boldsymbol{y} \mid \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{b},\ \boldsymbol{y} \geq \boldsymbol{0}^n\}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$. It is assumed that P is nonempty and bounded. Show that the problem can be written as a linear program as

$$\begin{aligned} \underset{\boldsymbol{x}, \boldsymbol{z}}{\text{maximize}} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{z} \leq \boldsymbol{x} \\ & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{z} \geq d \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \boldsymbol{z} \geq \boldsymbol{0}^m. \end{aligned}$$
