Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	15–08–27
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Åse Fahlander (0703-088304)
Result announced:	15–09–18 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 - x_2, \\
\text{subject to} \quad & x_1 + x_2 \geq 1, \\
& x_1 - 2x_2 \leq 1. \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{aligned}
$$

(2p) a) Solve the problem using phase I and phase II of the simplex method.
Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Does its LP dual have an optimal solution?

(3p) Question 2

(linear inequalities)

Consider the system of linear inequalities

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},$$

for which we assume there is at least one solution. Let d be a given scalar. Use linear programming duality to establish the equivalence of the following two statements:

(a) Every solution $\boldsymbol{x}$ to the system $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ satisfies $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq d$.

(b) There exists some vector $\boldsymbol{y} \geq \boldsymbol{0}$ such that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \leq d$.

(3p) Question 3

(the Frank–Wolfe algorithm)

Consider the problem to

$$
\begin{aligned}
\underset{x_1,x_2}{\text{minimize}} \quad & \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 52 & 34 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\
\text{subject to} \quad & x_1 + 2x_2 \le 4. \\
& x_1 + x_2 \le 3. \\
& 2x_1 \le 5. \\
& x_1 \ge 0. \\
& x_2 \ge 0.
\end{aligned}
\tag{1}
$$

Solve the problem (1) using the Frank–Wolfe algorithm. Start with the initial guess $\boldsymbol{x}^{(0)} = (x_1, x_2)^{\mathrm{T}} = (2.5, 0)^{\mathrm{T}}$. The line search should be performed as an exact minimization. If necessary, you are allowed to carry out the calculations approximately with two digits of accuracy.

Hint: You may find it helpful to analyze the problem and the algorithm progress graphically, but this must be augmented with a rigorous analysis.

(3p) Question 4

(modelling)

A small municipality is forced to close one or several schools. Out of ten existing schools, at most three schools can be closed. The annual cost to keep school i open is c_i kr. School i can educate a maximum of k_i students. The municipality is divided into J home areas and there is a requirement that all students in an area belong to the same school. There are b_j students in area j and the average distance from area j to school i is d_{ij} km. The estimated annual cost for student travels is set to m kr per km and student.

Formulate a linear integer program to decide on which schools to keep and which ones to close, such that we minimize the total cost for schools and travels and fulfill the above listed requirements.

Question 5

(true or false)

The below three claims should be assessed. Are they true or false? Provide an answer together with a short but complete motivation.

(1p) a) *Claim:* A strictly convex function is differentiable.

(1p) b) *Claim:* For a constrained minimization problem with explicit constraints, any Lagrangian dual formulation provides an upper bound on the optimal value of the original problem.

(1p) c) *Claim:* In linear programming, at termination of the Simplex method the optimal values of the dual variables are equal to the Lagrange multipliers of the linear constraints in the (primal) linear program.

Question 6

(interior penalty methods)

Consider the problem to

$$\begin{aligned} \text{minimize } & f(\boldsymbol{x}) := (x_1 - 2)^4 + (x_1 - 2x_2)^2, \\ \text{subject to } & g(\boldsymbol{x}) := x_1^2 - x_2 \leq 0. \end{aligned}$$

We attack this problem with an interior penalty (barrier) method, using the barrier function $\phi(s) = -s^{-1}$. The penalty problem is to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ f(\boldsymbol{x}) + \nu \hat{\chi}_S(\boldsymbol{x}), \tag{1}$$

where $\hat{\chi}_S(\boldsymbol{x}) = \phi(g(\boldsymbol{x}))$, for a sequence of positive, decreasing values of the penalty parameter ν.

We repeat a general convergence result for the interior penalty method below.

Theorem 1 (convergence of an interior point algorithm) *Let the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the functions g_i, $i = 1, \ldots, m$, defining the inequality* *constraints be in $C^1(\mathbb{R}^n)$. Further assume that the barrier function $\phi : \mathbb{R}_- \to \mathbb{R}_+$ is in C^1 and that $\phi'(s) \geq 0$ for all $s < 0$.*

Consider a sequence $\{\boldsymbol{x}_k\}$ of points that are stationary for the sequence of problems (1) with $\nu = \nu_k$, for some positive sequence of penalty parameters $\{\nu_k\}$ converging to 0. Assume that $\lim_{k\to+\infty} \boldsymbol{x}_k = \hat{\boldsymbol{x}}$, and that LICQ holds at $\hat{\boldsymbol{x}}$. Then, $\hat{\boldsymbol{x}}$ is a KKT point of the problem at hand.

In other words,

$$\left.\begin{array}{r} \boldsymbol{x}_k \text{ stationary in } (1) \\ \boldsymbol{x}_k \to \hat{\boldsymbol{x}} \text{ as } k \to +\infty \\ \text{LICQ holds at } \hat{\boldsymbol{x}} \end{array}\right\} \implies \hat{\boldsymbol{x}} \text{ stationary in our problem.}$$

(1p) a) Does the above theorem apply to the problem at hand and the selection of the penalty function?

(2p) b) Implementing the above-mentioned procedure, the first value of the penalty parameter was set to $\nu_0 = 10$, which is then divided by ten in each iteration, and the initial problem (1) was solved from the strictly feasible point $(0, 1)^{\mathrm{T}}$. The algorithm terminated after six iterations with the following results: $\boldsymbol{x}_6 \approx (0.94389, 0.89635)^{\mathrm{T}}$, and the multiplier estimate [given by $\nu_6\phi'(g(\boldsymbol{x}_6))$] $\hat{\mu}_6 \approx 3.385$. Confirm that the vector $\boldsymbol{x}_6$ is close to being a KKT point. Are the KKT point(s) globally optimal? Why/Why not?

Question 7

(the KKT conditions)

Consider the problem to

$$\begin{array}{ll} \text{minimize} & x_1x_2 + x_2x_3 + x_1x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 3. \end{array}$$

(2p) a) Write down the KKT conditions and find *all* KKT points.

(1p) b) Does the problem have an optimal solution? Motivate!
