Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	14–04–22
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points out of 3 Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions
Examiner:	Michael Patriksson
Teacher on duty:	Emil Gustavsson (0703-088304)
Result announced:	14–05–14 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$
\begin{aligned}
\text{minimize} \quad & z = 2x_1 + x_2, \\
\text{subject to} \quad & \phantom{z = {}} 2x_1 + x_2 \geq -2, \\
& \phantom{z = {}} 2x_1 + 5x_2 \leq 6, \\
& \phantom{z = {}2x_1 + 5} x_2 \geq 0.
\end{aligned}
$$

(2p) a) Solve the problem using the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables in the standard form.
Aid: Utilize the identity

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

(1p) b) Is the optimal solution obtained unique? Motivate your answer. If the optimal solution is not unique, then state all alternative optimal solutions.

(3p)

Question 2

(KKT conditions)

Let $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ and consider the optimization problem to

$$
\begin{aligned}
\text{minimize} \quad & -\log\left(\sum_{i=1}^{n} a_i x_i\right) - \log\left(\sum_{i=1}^{n} \frac{x_i}{a_i}\right) \\
\text{subject to} \quad & \sum_{i=1}^{n} x_i = 1 \\
& \quad x_i \geq 0,\ i = 1, \ldots, n.
\end{aligned}
$$

Show that $\mathbf{x} = (1/2, 0, \ldots, 0, 1/2)^{\mathrm{T}}$ is an optimal solution.

Question 3

(problem decomposition)

Consider the problem to minimize a convex and differentiable function f of the form

$$f(\boldsymbol{x}) := \sum_{i \in \mathcal{I}} f_i(\boldsymbol{x}_i),$$

where $\mathcal{I}$ is a finite index set and $\boldsymbol{x}_i \in \mathbb{R}^n$, subject to two types of constraints: (1) an individual feasible set for each "block" of variables $\boldsymbol{x}_i$, of the form

$$\boldsymbol{x}_i \in X_i, \qquad i \in \mathcal{I},$$

where the sets X_i are non-empty polyhedral sets in $\mathbb{R}^n$, and (2) a total resource constraint of the form

$$\sum_{i \in \mathcal{I}} \boldsymbol{x}_i \leq \boldsymbol{u},$$

for some vector $\boldsymbol{u} \in \mathbb{R}^n$.

(2p) a) Describe how a Lagrangian relaxation algorithm for this problem would appear if we Lagrangian relax the resource constraint. Describe in particular the appearance of the Lagrangian subproblem, and how you would solve it. Can you easily provide a lower bound on the optimal value of the original problem? How?

(1p) b) Suppose next that f is quadratic and that $n_i = 1$ for all $i \in \mathcal{I}$. Describe how the Lagrangian subproblem can be solved analytically.

(3p) Question 4

(Frank-Wolfe algorithm)

Consider the problem to

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimize}} \quad & f(\boldsymbol{x}) := \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 52 & 34 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\
\text{subject to} \quad & x_1 + 2x_2 \leq 4, \\
& x_1 + x_2 \leq 3, \\
& 2x_1 \leq 5, \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{aligned} \tag{1}$$

Solve problem (1) with the Frank–Wolfe algorithm. Start with initial guess $\boldsymbol{x}^{(0)} = (x_1, x_2)^{\mathrm{T}} = (2.5, 0)^{\mathrm{T}}$. Use exact minimization for line search. If necessary, you are allowed to carry out the calculations approximately with two digits of accuracy.

[*Hint:* You may find it helpful to analyze the problem and the algorithm progress in a picture, but this should be augmented with rigorous analysis.]

Question 5

(true or false)

The below three claims should be assessed. Are they true or false? Provide an answer together with a short but complete motivation.

(1p) a) Suppose that you wish to solve a linear integer program, and that you start by solving its continuous relaxation. Suppose that $\bar{\boldsymbol{x}}$ is a solution to this problem. Then, an optimal solution to the integer program can always be found by rounding, individually, each element of $\bar{\boldsymbol{x}}$ either up or down to the nearest integer value.

(1p) b) Suppose that you are able to solve a nonlinear optimization problem and that in a globally optimal solution to it there is one inequality constraint that is satisfied with *strict* inequality. Then, this inequality is redundant, and can be removed without affecting the optimal solution.

(1p) c) Suppose that you consider minimizing a convex and differentiable function f over a closed convex set S, and that you have found an optimal solution

$\boldsymbol{x}^*$. Suppose also that there is another optimal solution $\bar{\boldsymbol{x}}$. Then, all points on the line segment between $\boldsymbol{x}^*$ and $\bar{\boldsymbol{x}}$ must also be optimal.

(3p) Question 6

(the Relaxation Theorem)

Given the problem to find

$$f^* := \operatorname*{infimum}_{x} \; f(\boldsymbol{x}), \tag{1a}$$

$$\text{subject to } \boldsymbol{x} \in S, \tag{1b}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given function and $S \subseteq \mathbb{R}^n$, we define a *relaxation* to (1) to be a problem of the following form: find

$$f_R^* := \operatorname*{infimum}_{x} \; f_R(\boldsymbol{x}), \tag{2a}$$

$$\text{subject to } \boldsymbol{x} \in S_R, \tag{2b}$$

where $f_R : \mathbb{R}^n \to \mathbb{R}$ is a function with the property that $f_R \leq f$ on S, and where $S_R \supseteq S$. For this pair of problems, we have the following basic result. You are asked to establish it.

Theorem 1 (Relaxation Theorem)

(a) [relaxation] $f_R^* \leq f^*$.

(b) [infeasibility] *If* (2) *is infeasible, then so is* (1).

(c) [optimal relaxation] *If the problem* (2) *has an optimal solution,* $\boldsymbol{x}_R^*$, *for which it holds that*

$$\boldsymbol{x}_R^* \in S \qquad \text{and} \qquad f_R(\boldsymbol{x}_R^*) = f(\boldsymbol{x}_R^*), \tag{3}$$

then $\boldsymbol{x}_R^*$ *is an optimal solution to* (1) *as well.*

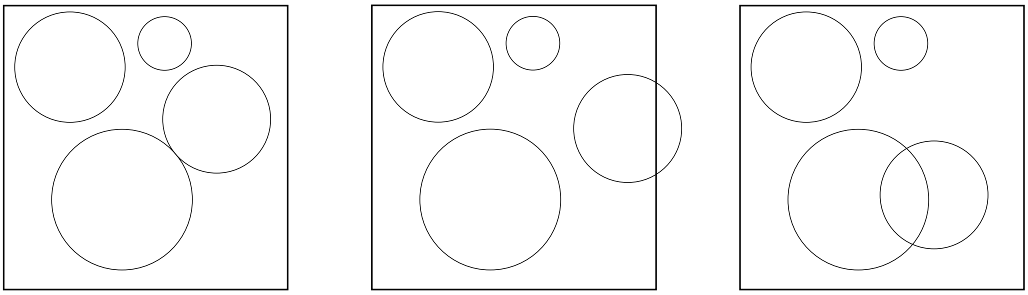
Question 7

(modelling)

Consider a square with side length L and corners in $(0, 0)$, $(0, L)$, $(L, 0)$ and (L, L). Formulate the problem of placing n circles inside the square in such a way that no circles overlap and such that the total area covered by the circles is maximized. Note that the radius of each circle should also be a variable in the optimization model.

Tip: Let r_i be the radius of circle $i = 1, \dots, n$, and let (x_i, y_i) be the coordinate of the center point of circle $i = 1, \dots, n$. These are the only variables you will need in the optimization model.

In the figures below you can see one feasible solution and two infeasible solutions for the problem when $n = 4$.



Feasible	Infeasible (not inside the square)	Infeasible (overlapping)