Chalmers/GU
Mathematics

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	13–12–17
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Zuzana Sabartova (0703-088304)
Result announced:	14–01–13
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & z = 3x_1 - x_2 + x_3, \\ \text{subject to} \quad & x_1 + 3x_2 - x_3 \leq 5, \\ & -2x_1 + x_2 - 2x_3 \leq -2, \\ & x_1, \quad x_2, \quad x_3 \geq 0. \end{aligned}$$

(2p) a) Solve the problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables in the standard form. Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Suppose that to the original problem we add a new variable x_4 and obtain the new problem to

$$\begin{aligned} \text{minimize} \quad & z = 3x_1 - x_2 + x_3 - \frac{1}{2}x_4, \\ \text{subject to} \quad & x_1 + 3x_2 - x_3 + 8x_4 \leq 5, \\ & -2x_1 + x_2 - 2x_3 - x_4 \leq -2, \\ & x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0. \end{aligned}$$

If the original problem has an optimal solution, explain how the optimal solution is affected by adding the new variable. If the original problem is unbounded, investigate if adding the new variable affects the unboundedness of the problem.

Note: Use your calculations from a) to answer the question.

Question 2

(nonlinear programming)

(1p) a) Consider the function $f(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$, where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^2$. At $\boldsymbol{x} = (-3, 4)^{\mathrm{T}}$, which directions $\boldsymbol{p} \in \mathbb{R}^2$ are descent directions with respect to f?

(2p) b) Consider the problem of minimizing the function $f(x) := x^2$ subject to the constraint that $x \geq 1$. Consider an extension of the standard exterior penalty method for this problem, in which the penalty function is

$$F_k(x) := \begin{cases} k(1-x), & x < 1, \\ 0, & x \geq 1, \end{cases}$$

where k is a non-negative integer. Derive the solution for this penalized problem for any positive value of the parameter k, and show that this penalty function yields convergence to the optimal solution for a *finite* value of the parameter.

(3p) Question 3

(characterization of convexity in C^1)

Let $f \in C^1$ on an open convex set S. Establish the following characterization of the convexity of f on S:

f is convex on $S \Longleftrightarrow f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x})$, for all $\boldsymbol{x}, \boldsymbol{y} \in S$.

Question 4

(modelling)

A company serving as a middle hand wants to plan its inventory management of a product over the next week. Each day $t = 1, \ldots, 7$, the company can buy the product from a producer to a cost of c_t per unit. The demand of the product the company needs to fulfill on day t is d_t, where $t = 1, \ldots, 7$. The company has a storage facility where it can store at most M units of the product at a cost of g per unit and day. On the first day ($t = 1$) the storage facility is empty.

(2p) a) Formulate a linear optimization model for the minimization of the cost for buying the product, while fulfilling the demand at each time step. Note that it is possible to buy fractions of units of the product.

(1p) b) The producer has realized that the company sometimes purchase large quantities on certain days. Therefore, the producer has decided that each day the company buys more than K units, the company needs to pay more. For units purchased over the limit K on day t, the the company pays c_t^{high} per unit, where $c_t^{\text{high}} > c_t$.

(Example: Let $t = 1$ and $K = 10$. If the company buys 12 units on day $t = 1$, then the cost is $10c_1 + 2c_1^{\text{high}}$)

Extend the model in a) such that the new information is taken into account. Note that the model should still be a linear optimization model, i.e., no binary variables.

Question 5

(true or false)

The below three claims should be assessed. Are they true or false? Provide an answer together with a short but complete motivation.

(1p) a) Let $\boldsymbol{p} \neq \mathbf{0}^n$ be a subgradient to the convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$ at the point $\boldsymbol{x} \in \mathbb{R}^n$.

Claim: $-\boldsymbol{p}$ is a descent direction to f at $\boldsymbol{x}$.

(1p) b) Consider the problem to

$$\begin{aligned} \text{minimize} \quad & 0, && (1a) \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, m, && (1b) \\ & \boldsymbol{x} \in X, && (1c) \end{aligned}$$

where $X \subseteq \mathbb{R}^n$. Let $q : \mathbb{R}^m \mapsto \mathbb{R}$ be the dual function obtained by Lagrangian relaxing the constraints (1b). We have that at a point $\bar{\boldsymbol{u}} \geq \mathbf{0}^m$, $q(\bar{\boldsymbol{u}}) = 1$.

Claim: The set $\{\boldsymbol{x} \in X \mid g_i(\boldsymbol{x}) \leq 0,\ i = 1, \dots, m\}$ is empty.

(1p) c) Suppose $S \subseteq \mathbb{R}^n$ is a nonempty and convex set, and let $f \in C^1$ on $\mathbb{R}^n$. Define the function $F : \mathbb{R}^n \mapsto \mathbb{R} \cup \{-\infty\}$ by

$$F(\boldsymbol{x}) := \operatorname*{infimum}_{\boldsymbol{y} \in S} \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}).$$

Claim: $F(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in S$.

Question 6

(KKT conditions)

Consider the problem

$$\begin{aligned}
\text{minimize} \quad & -(x_1-2)^2-(x_2-2)^2, && \text{(1a)}\\
\text{subject to} \quad & (x_1+x_2-4)^2 \geq 1, && \text{(1b)}\\
& 0 \leq x_1 \leq 4, && \text{(1c)}\\
& 0 \leq x_2 \leq 4. && \text{(1d)}
\end{aligned}$$

(2p) a) Find all KKT-points. (You may do this graphically.)

(1p) b) Motivate logically why the problem (1) has an optimal solution among the KKT-points.

Question 7

(linear programming duality and optimality)

Consider a project management problem whose decision variables are the starting times (i.e., t_1, t_2, t_3, t_4) of four different tasks. Each task requires a given amount of time to complete (i.e., $T_i \geq 0$ for $i = 1, 2, 3, 4$ given). In addition, between certain tasks there can be precedence constraints summarized in Figure 1. Specifically, if in Figure 1 there is an edge from node i to node j, then it means that task j cannot start before task i is completed (i.e., $t_j \geq t_i + T_i$). The objective of the project management problem is to minimize the total duration of the project involving the four tasks. The objective function is $t_4 + T_4 - t_1$, but the constant T_4 can be removed from the objective. In summary, the project management problem can be modeled as

$$\begin{aligned} \underset{t_1,t_2,t_3,t_4}{\text{minimize}} \quad & t_4 - t_1 \\ \text{subject to} \quad & t_2 - t_1 \geq T_1, \\ & t_3 - t_1 \geq T_1, \\ & t_3 - t_2 \geq T_2, \\ & t_4 - t_2 \geq T_2, \\ & t_4 - t_3 \geq T_3, \end{aligned}$$

and we will call this the primal problem.

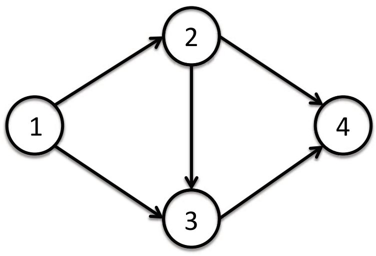

Figure 1: The graph describing the precedence constraints between the tasks in the project management problem. A directed edge from node i to node j means that task j cannot start before task i is completed (e.g., $i = 1$ and $j = 2$).

(1p) a) Derive the dual linear program (i.e., the dual problem), with one dual decision variable for each primal precedence constraint.

(1p) b) Specialize the problem data to $T_1 = 1$, $T_2 = 2$, $T_3 = 1$ and $T_4 = 1$. Suppose by inspection, we obtain the primal optimal solution as

$$t_1^* \text{ free}, \quad t_2^* = t_1^* + 1, \quad t_3^* = t_1^* + 3, \quad t_4^* = t_1^* + 4.$$

What is the optimal solution to the dual problem?

Hint: The following provides an idea of how to approach the solution, but it is not required that the given idea is followed. Consider putting weight T_i to each edge from node i to node j in Figure 1. Which is the directed path from node 1 to node 4 with the maximum sum of edge weights? Does this give you a feasible solution to the dual problem? How do you certify the optimality of the dual feasible solution?

(1p) c) Verify that the complementary slackness conditions indeed hold for the primal and dual optimal solutions.