Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date: 11–04–26
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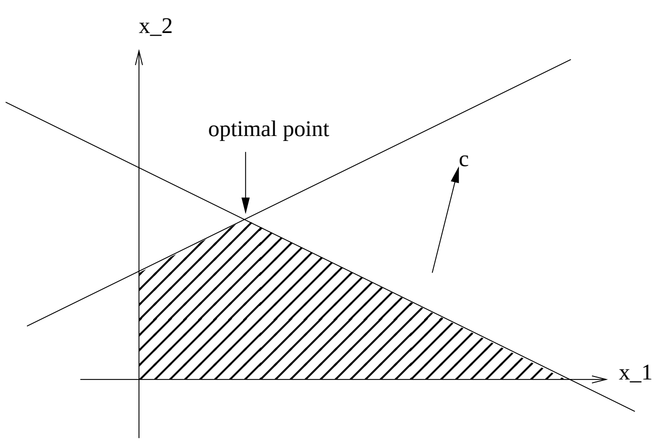
(3p) Question 1

(the simplex method)

(2p) a) We first rewrite the problem on standard form. We multiply the objective by -1 to obtain a minimization problem, multiply one of the constraints by -1 to obtain a non-negative r.h.s. and introduce the slack variables s_1 and s_2. We obtain

$$\begin{array}{llrrrrl} \text{minimize} & z = & -x_1 & -4x_2 & & & \\ \text{subject to} & & x_1 & +2x_2 & +s_1 & & = 4 \\ & & -x_1 & +2x_2 & & +s_2 & = 2 \\ & & x_1, & x_2, & s_1, & s_2 & \geq 0. \end{array}$$

An obvious starting BFS is (s_1, s_2) and we can thus begin with phase II. The vector of reduced costs for x_1 and x_2 yields $(-1, -4)$. Thus x_2 enters the basis. The minimum ratio tests shows that s_2 leaves the basis. The new BFS is thus (s_1, x_2). The reduced costs for x_1 and s_2 are $(-3, 2)$. Thus x_1 enters the basis and minimum ratio test yields that s_1 leaves the basis. The new BFS is (x_1, x_2) and the reduced costs corresponding to s_1 and s_2 are $(3/2, 1/2)$. Hence the solution $(x_1, x_2) = B^{-1}b = (1, 3/2)$ is optimal. We can see that the calculations are correct by drawing a picture.



(1p) b) For the BFS to be optimal, the reduced costs must fulfill

$$c_N^T - c_B^T B^{-1} N \geq 0.$$

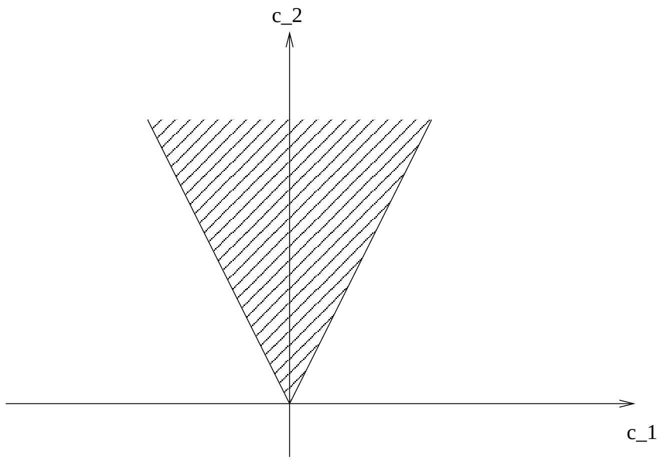
Since $N = I$,

$$B^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/4 & 1/4 \end{pmatrix},$$

$c_N = (0,0)$ and $c_B = (-c_1, -c_2)$, we obtain

$$\begin{aligned} 1/2c_1 + 1/4c_2 &\geq 0, \\ -1/2c_2 + 1/4c_2 &\geq 0. \end{aligned}$$

The same conclusion can be drawn from the KKT conditions. Drawing the region we obtain:



(3p) ## Question 2

(the Separation Theorem)

See Theorem 4.28 in The Book.

Question 3

(descent and optimality in optimization)

(1p) a) At $\boldsymbol{x} = (1,1)^{\mathrm{T}}$, $\nabla f(\boldsymbol{x}) = (1,14)^{\mathrm{T}}$, so $\nabla f(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{p} = (1,14)(1,-2)^{\mathrm{T}} = -27 < 0$. Yes, the vector $\boldsymbol{p}$ is a vector of descent.

(2p) b) As the only two constraints are affine (in fact linear) Abadie's CQ is fulfilled at every feasible point. Hence, at the local minimum $\boldsymbol{x}^* = \boldsymbol{0}^2$, the KKT conditions must be fulfilled. The two constraints have non-negative multipliers, μ_1 and μ_2; as the two inequalities are fulfilled with equality at $\boldsymbol{x}^*$, complementarity slackness is fulfilled even if $\mu_i > 0$ for any of $i = 1, 2,$

so the only requirements sofar are that $\mu_i \geq 0$, $i = 1, 2$. What remains is to study the requirements from dual feasibility—the first row of the KKT conditions. From the requirement that $\nabla f(\boldsymbol{x}^*) + \boldsymbol{\mu}^{\mathrm{T}} \nabla g(\boldsymbol{x}^*) = \mathbf{0}^2$ we get that

$$\nabla f(\boldsymbol{x}^*) + \mu_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{0}^2,$$

so we conclude that the value of $\nabla f(\boldsymbol{x}^*)$ is of the form

$$\nabla f(\boldsymbol{x}^*) = \mu_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

for any non-negative values of $\boldsymbol{\mu} \geq \mathbf{0}^2$. (This is the cone spanned by the active constraints at $\boldsymbol{x}^*$, or, in other words, the normal cone to the faesible set at $\boldsymbol{x}^*$.)

(3p) Question 4

(optimality conditions) We rewrite the problem as that to minimize $f(\boldsymbol{x}) := -\boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}$, in order to fit with the standard formulation of the optimality conditions. The problem is now convex, and its only constraint has an interior point, so Slater's CQ is fulfilled. This means that the KKT conditions are necessary as well as sufficient for a global optimal solution.

The solution suggested, $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$, fulfills the only constraint with equality, whence the Lagrange multiplier may be positive. The first row of the KKT conditions then states that

$$-\boldsymbol{b} + 2\mu^* \boldsymbol{x}^* = \mathbf{0}^n,$$

that is,

$$-\boldsymbol{b} + 2\mu^* \boldsymbol{b}/\|\boldsymbol{b}\| = \mathbf{0}^n.$$

With the identification $\mu^* = \|\boldsymbol{b}\|/2$, we verify that $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$ fulfills the KKT conditions. As noted above, the problem is a convex one, so $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$ is indeed the unique globally optimal solution to the problem.

Question 5

(modeling)

(2p) a) Introduce the variables x_{jt} for the amount imported from producer j between time t and $t+1$. Let y_t be the amount stored between time t and $t+1$ and let $y_0 = 0$. The model is to

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=0}^{T-1} \left(\sum_{j=1}^{n} c_{jt} x_{jt} + f y_t \right), \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{jt} + y_t - y_{t+1} = d_{t+1}, \quad t = 0, \dots, T-1, \\
& y_0 = 0, \\
& y_t \leq M, \quad t = 1, \dots, T, \\
& x_{jt} \leq k_{jt}, \quad t = 0, \dots, T-1, \\
& y_t \geq 0, \quad t = 1, \dots, T, \\
& x_{jt} \geq 0, \quad j = 1, \dots, n, \ t = 0, \dots, T.
\end{aligned}
$$

(1p) b) Introduce z_t as the shortage at time step t. The model is to

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=0}^{T-1} \left(\sum_{j=1}^{n} c_{jt} x_{jt} + f y_t + \gamma z_t \right), \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{jt} + y_t - y_{t+1} + z_{t+1} = d_{t+1}, \quad t = 0, \dots, T-1, \\
& y_0 = 0, \\
& y_t \leq M, \quad t = 0, \dots, T, \\
& x_{jt} \leq k_{jt}, \quad t = 0, \dots, T-1, \\
& y_t \geq 0, \quad t = 1, \dots, T, \\
& z_t \geq 0, \quad t = 1, \dots, T, \\
& x_{jt} \geq 0, \quad j = 1, \dots, n, \ t = 0, \dots, T.
\end{aligned}
$$

(3p) Question 6

(the gradient projection algorithm)

Iteration 1: We have $\nabla f(\boldsymbol{x}^0) = (-2, -3)^{\mathrm{T}}$. We need to project the point $(0, 0)^{\mathrm{T}} - (-2, -3)^{\mathrm{T}} = (2, 3)^{\mathrm{T}}$ on the feasible region X. From the figure, we see that $\mathrm{Proj}_{\boldsymbol{x} \in X} \left((2, 3)^{\mathrm{T}} \right) = (2, 2)^{\mathrm{T}}$. Hence, $\boldsymbol{x}^1 = (2, 2)^{\mathrm{T}}$.

Iteration 2: We have $\nabla f(\boldsymbol{x}^1) = (-2, 1)^{\mathrm{T}}$. We need to project the point $(2, 2)^{\mathrm{T}} -$

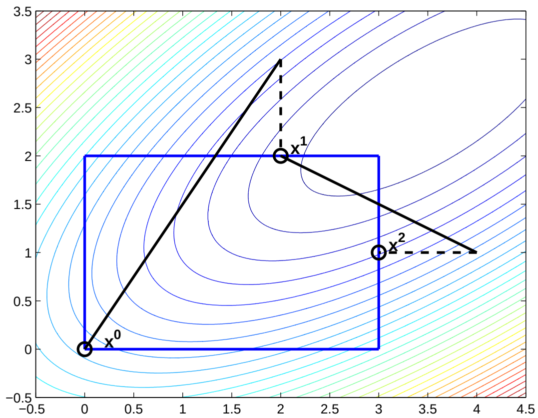


Figure 1: Path taken by the gradient projection algorithm.

$(-2, 1)^{\mathrm{T}} = (4, 1)^{\mathrm{T}}$ on the feasible region X. From the figure, we see that $\mathrm{Proj}_{\boldsymbol{x}\in X}\left((4,1)^{\mathrm{T}}\right) = (3,1)^{\mathrm{T}}$. Hence, $\boldsymbol{x}^2 = (3,1)^{\mathrm{T}}$.

To check if $\boldsymbol{x}^2$ is a global/local minima, we consider the KKT conditions. All constraints are convex and there exists an inner point, which implies that Slater's constraint qualification holds. Hence, the KKT conditions are necessary for optimality. We can also note that the objective function is convex, which implies that the KKT conditions are also sufficient for optimality. At $\boldsymbol{x}^2$, the only active constraint is $g(\boldsymbol{x}) := x_1 - 3$, and we have $\nabla g(\boldsymbol{x}) = (1,0)^{\mathrm{T}}$. Since $\nabla f(\boldsymbol{x}^2) = (2, -5)$, we note that $\nabla f(\boldsymbol{x}) \neq \mu \nabla g(\boldsymbol{x})$ for any positive μ, and hence $\boldsymbol{x}^2$ is not a KKT point, and is therefore neither a global nor local minima.

Question 7

(short questions)

(1p) a) No! Consider the problem to

$$\underset{x\in[-1,2]}{\text{maximize}}\, x^2;$$

the point $x = -1$ is a KKT point, but it is only a local maximum (not a global). Further, $x = 0$ is also a KKT point, but it is not even a local maximum.

(2p) b) According to Theorem 6.4 (p. 144 in the course book) the dual function is always concave. Hence the dual problem (which we maximize) is a convex problem, which in turn implies that KKT is sufficient for optimality. Thus we can conclude that $\hat{\mu}$ is an optimal solution to the dual problem. Weak duality implies that we have a lower bound on the primal problem, that is, we know that $f(x^*) \geq q(\hat{\mu})$. If the set X is convex and f is convex, we have a convex problem (as the two additional constraints are also convex). Further, an interior point exists with respect to the constraints. We can therefore conclude (from Theorem 6.9 on page 149) that no duality gap exists. Hence $f(x^*) = q(\hat{\mu})$.