1. Consider the LP:

$$\max_{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4} 6x_1 - 3x_2 - 2x_3 + 5x_4$$
$$\text{subject to}$$
$$4x_1 + 3x_2 - 8x_3 + 7x_4 = 11$$
$$3x_1 + 2x_2 + 7x_3 + 6x_4 \geq 23$$
$$7x_1 + 4x_2 + 3x_3 + 2x_4 \leq 12$$
$$x_1, x_2 \geq 0$$
$$x_3 \leq 0 \tag{1}$$

   (a) Write the dual LP for the LP in (1).

   (b) Write the Lagrangian dual function for the optimization in (1).

   (c) Write the Lagrangian dual optimization of (1) and show that the Lagrangian dual optimization is identical to the dual LP obtained in part (a).

2. A company plans to establish a communication network between its four workstations, indexed by a, b, c, d. A network is a set of links between individual workstations. The links are undirected i.e., the link between workstations i and j can be used to communicate from both i to j and j to i. The cost of establishing link between different workstations is given in Table 1, where ∞ denotes "impossible" link. Notice that only the upper triangular part of the table is required and the lower triangular part is just a copy of the upper one.

Indexes	a	b	c	d
a	∞	1	∞	3
b	1	∞	1	3
c	∞	1	∞	2
d	3	3	2	∞

Table 1: The cost of individual links in Question 2.

The company is to design a network, which satisfies the following two conditions:

- Connectivity: It is possible to communicate between any two work stations.
- Optimality: The cost of establishing the network is minimized.

   (a) Define incidence variables $x_{ij} \in \{0, 1\}$ for every "possible" link, where $x_{ij} = 1$ means that the link between the workstations i and j is selected. According to these variables, write an ILP, which formulates the two items above. (Hint: You may use the cut-set constraints to ensure connectivity.)

(b) Prove that even if the cost values in Table 1 change, as long as they are positive (and not zero), the solution to the ILP in part (a) (the optimal network) does not have a cycle (i.e., its graph is a tree).

(c) Obtain a new ILP by adding a linear **equality** constraint to the ILP in part (a), which guarantees that the solution is a tree. (Hint: what is the number of edges in a tree?).

(d) Solve the LP relaxation of the new ILP by CVX and plot the resulting network as a graph, where the nodes are the workstations and the edges are the selected links. Hand in the CVX code as a part of your solution. (round the solution to the nearest feasible integer point if necessary).

(e) What is the problem with the approach in this question using CVX for large graphs?

3. For the construction of a new bridge a financing plan has to be established. Table 2 gives the estimated cost over the 6 years of construction. The city

Year	Cost
1	MSek 20
2	MSek 17
3	MSek 23
4	MSek 24
5	MSek 25
6	MSek 21

Table 2: estimated cost over the 6 years

plans to raise the funds needed to pay these costs by issuing bonds. Each bond is issued on 1st of January of a year $i \in \{1, 2, \dots, 6\}$ and is due on the 31st December of another year $j \in \{i, i+1, \dots, 6\}$. The validity period of each bound is fixed and is decided when issued. According to Table 3, interest has to be paid on bonds only once when they are due, depending on how long they are valid. Money that is not used for construction can be

Period, year(s)	Interest rate
1	7%
2	15%
3	23%
4	32%
5	41%
6	50%

Table 3: Total interest rate for different terms

invested at a bank at an interest rate of $r = 6.8\%$ annually. The problem is to find out how many bonds to which terms should be issued each year to keep the outstanding debts on the 31st December in year 6 as low as possible (1 bond=1 SEK).

Take x_{ij} as the number of bonds issued on 1st of January in year $i = 1, 2, \ldots, 6$ and is due on 31st December in year $j = i, i+1, \ldots, 6$. Assume that x_{ij} is continuous. Write a linear program that models the problem and use CVX to give an optimal solution and the optimal value to the problem. Hand in your CVX code.

4. Consider the following LP relaxation of the $0-1$ knapsack problem:

$$\begin{aligned} \max_{\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n} \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{subject to} \quad & \\ & a_1x_1 + a_2x_2 + \ldots + a_nx_n \le b \\ & 0 \le x_i \le 1 \quad i = 1, 2, \ldots, n \end{aligned} \tag{2}$$

where $b > 0$, $a_i > 0, , c_i > 0$ for $i = 1, 2, \ldots, n$, and

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \ldots \ge \frac{c_n}{a_n} \tag{3}$$

(a) Write the dual LP of this problem.

(b) Using the complementary slackness conditions show that exactly one of the following two conditions occurs:

i. $\sum_{j=1}^{n} a_j \le b$: Then, the solution is given by $x_j = 1$ for $j = 1, 2, \ldots, n$.

ii. $\sum_{j=1}^{n} a_j > b$: Take the smallest value for the variable $r \in \{1, 2, \ldots, n\}$ such that

$$\sum_{j=1}^{r} a_j > b \tag{4}$$

Then, the solution is given by

$$x_j = \begin{cases} 1 & j < r \\ \frac{b - \sum_{j=1}^{r-1} a_j}{a_r} & j = r \\ 0 & j > r \end{cases} \tag{5}$$

5. A large manufacturing company decides to construct a number of storage facilities, in order to supply its different factories. After a careful investigation n different locations are selected as candidates. Denote the factories by $F_1, F_2, ..., F_m$, where m is the number of factories. Due to physical limitations, each storage location may supply only a subset of

the factories. Denote by $S_k \subseteq \{F_1, F_2, \ldots, F_m\}$ the subset of plants that the storage location k may cover. Also denote by c_k the cost of constructing a storage facility at the k^{th} location. It is important to ensure that each factory is supplied by at least one facility. Then, the problem is to select a number of candidate locations, which minimize the construction cost.

(a) By assigning a binary variable $x_k \in \{0, 1\}$ to each candidate location, formulate an ILP which solves the problem.

(b) Write the LP relaxation and its dual.

(c) Devise a primal-dual algorithm to approximately solve this problem: Compare the LP and its dual here with the one for the minimum vertex cover problem given in the lecture notes, Example 21. Make necessary changes in the minimum vertex cover algorithm.

6. Consider the optimization

$$\min_{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n} \sum_{i=1}^{n} \frac{1}{2\mu_i}(x_i - \bar{x}_i)^2 + \sum_{i=1}^{n} \lambda_i |x_i| + \sum_{i=1}^{n} x_i g_i \tag{6}$$

where $\mu_i, \lambda_i > 0$ and $\bar{x}_i, g_i$ are real constants. Show that the point

$$x_i = \mathcal{T}_{\mu_i \lambda_i}\left(\bar{x}_i - \mu_i g_i\right) \quad i = 1, 2, \ldots, n \tag{7}$$

is an optimal solution, where

$$\mathcal{T}_\tau(y) = \begin{cases} y - \tau & y \geq \tau \\ y + \tau & y \leq -\tau \\ 0 & -\tau < y < \tau \end{cases} \tag{8}$$

The function $\mathcal{T}_\tau(y)$ is called the *shrinkage function*.