Discrete Optimization Take Home Exam [1]

March 12	Take Home, Due March 13, 10 AM

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Poäng:	60	
Betygsgränser:	Chalmers	5:48, 4:36, 3:24
	GU	VG:48, G:28
	Doktorander	G:36
Hjälpmedel:	Course book, material on course page.	

- You are required to **work alone**.
- Recommended: First look through all questions and make sure that you understand them properly. In case of doubt, do not hesitate to ask.
- Answer concisely and to the point. (English if you can and Swedish if you must!)
- Code strictly forbidden! Motivated pseudocode or plain but clear English/Swedish description is fine.

Lycka till!

[1] 2009 läsperiod 3, TDA206/DIT370.

Problem 1 Bourbon and Whiskey [10] *Chalmers Liquids Inc.* (CLI) is engaged in the production and sale of two kinds of hard liquor. CLI purchases intermediate-stage products in bulk, purifies them by repeated distillation, mixes them, bottles the product under its own brand names and sells it. One product is a bourbon, the other a blended whiskey. The problem is to decide how many bottles of each should be produced in the next production period. As the companys products are very popular on the market, the production capacity is inadequate to produce all that CLI might sell. The bourbon requires 3 machine hours per bottle, while the blended whiskey requires 4 hours of machine time per bottle. There are 20,000 machine hours available in the production period. The direct operating costs, which are mainly for labour and materials, are SEK 30.00 per bottle of bourbon and SEK 20.00 per bottle of blended whiskey. The working capital available to finance these costs is SEK 44000; however, 40% of the sales revenues will be collected during the production period and made available to finance ongoing operations. The selling price is SEK 55 for a bottle of bourbon and SEK 45 for a bottle of blended whiskey.

(a) Set up a linear program in two variables x_1 and x_2 that maximizes CLI's profit in the production period to come, subject to limitations on machine capacity and working capital.

(b) Sketch the set of feasible solutions in the plane and give the coordinates of the vertices.

(c) What is the optimal production mix to schedule and how large is the company's profit with this schedule?

(d) Suppose CLI could spend some money to repair machinery and increase its available machine hours by 2000 hours (before production starts). Should the investment be made and if so, up to which price? Hint: How does this change affect the linear program and your sketch?

Problem 2 Cliques [10] The *max clique* problems is as follows: given a graph $G = (V, E)$ with weights $w_v \geq 0, v \in V$ on the vertices, find the subset $U \subseteq V$ of macimum total weight $w(U) := \sum_{u \in U} w_u$ such that all vertices in U are connected to each other. The problem is NP-hard in general, but show that if G is a bipartite graph, then the problem can be solved in polynomial time.

Problem 3 Simplex [10] Show that the following linear program is unbounded below and find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} + \lambda\mathbf{v}$ is feasible for all $\lambda \geq 0$. Find a feasible point with value -98.

$$\min \quad -2x_1 - 3x_2 + x_3$$

subject to:

$$\begin{aligned} x_1 + x_2 + x_3 &\geq -3 \\ -x_1 + x_2 - x_3 &\geq -4 \\ x_1 - x_2 - 2x_3 &\geq -1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Problem 4 LP rounding [10] In the set cover problem, we are given subsets $S_1, S_2, \cdots, S_m$ of a universal set U with associated costs $c_1, c_2, \cdots, c_m \geq 0$. (Think of the universal set as aircraft routes to be covered, and each subset as possible routes that a certain class of aircraft in a fleet, asy the 737s can cover). The problem is to select some of the subsets $S_{i_1}, S_{i_2}, ..., S_{i_k}$ such that the sum $\sum_\ell c_{i_\ell}$ is minimized while the union of the selected sets $\bigcup_\ell S_{i_\ell} = U$ i.e. the selected subsets cover all the elemnts of the universal set.

(a) Show that the vertex cover problem is a special case of the set cover problem.

(b) Formulate the set cover problem as a ILP.

(c) Give an algorithm based on deterministically rounding the LP relaxation.

(d) Analyse the approximation guarantee of the resulting algorithm in terms of the parameters of the problem input (in the special case of the vertex cover problem, you should recover the approximation factor of 2.)

Problem 5 Primal Dual [10] In the set cover problem, we are given subsets $S_1, S_2, \cdots, S_m$ of a universal set U with associated costs $c_1, c_2, \cdots, c_m \geq 0$. (Think of the universal set as aircraft routes to be covered, and each subset as possible routes that a certain class of aircraft in a fleet, asy the 737s can cover). The problem is to select some of the subsets $S_{i_1}, S_{i_2}, ..., S_{i_k}$ such that the sum $\sum_\ell c_{i_\ell}$ is minimized while the union of the selected sets $\bigcup_\ell S_{i_\ell} = U$ i.e. the selected subsets cover all the elemnts of the universal set.

(a) Formulate the problem as an ILP and pass to the LP relaxation.

(b) Write the dual of the LP, and give an economic interpretation of the dual variables.

(c) Write down the complementary slackness conditions for optimality of the primal and dual LPs.

(d) Develop a primal-dual algorithm for the problem guided by the complementary slackness conditions. First give the continuous time version and then describe briefly how you'd actually implement it.

(e) Give an analyis of the approximation guarantee of the algorithm (in the special case of the vertex cover, you should recover the approximation factor of 2).

Problem 6 Primal Dual [10] Consider the facility location problem with n possible locations for the facilities with opening costs $c_1, \cdots, c_n$, and m customers with metric distances $d_{i,j}$ between facility i and customer j, $i \in [n], j \in [m]$. In addition, there is a *budget limit* B on the total cost of facilities that can be opened.

(a) Formulate a ILP for the problem and write its LP relaxation.

(b) Formulate an appropriate Lagrangian relaxation of the LP.

(c) Construct the dual to the Lagrangian relaxation.

(d) Suggest a primal-dual based algorithm for the problem.

(e) Outline an analysis of the approximation performance of the algorithm.