Solutions to SSY285 240830

7) Volume balances for the two tanks give

$$\frac{d}{dt}(A_1 h_1) = q_1 + q_0 - q_{12}$$

$$\frac{d}{dt}(A_2 h_2) = q_{12} - q_2$$

which gives the nonlinear state equations

$$\dot{h}_1 = \frac{1}{A_1} u + \frac{1}{A_1} q_0 - \frac{k_{12}}{A_1}\sqrt{h_1 - h_2} = f_1(h, u, q_0)$$

$$\dot{h}_2 = \frac{k_{12}}{A_2}\sqrt{h_1 - h_2} - k_2\sqrt{h_2} = f_2(\cdot\ \cdot\ )$$

a) Linearization around $\bar{h}_1 = 2$ and $\bar{h}_1 = 1$

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial h_1} & \frac{\partial f_1}{\partial h_2} \\ \frac{\partial f_2}{\partial h_1} & \frac{\partial f_2}{\partial h_2} \end{bmatrix}_{(\bar{h},\bar{u})}}_{A} \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}}_{B} \Delta u + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial q_0} \\ \frac{\partial f_2}{\partial q_0} \end{bmatrix}}_{N} \Delta q_0$$

$$A = \begin{bmatrix} \frac{-k_{12}}{2A_1\sqrt{\bar{h}_1 - \bar{h}_2}} & \frac{k_{12}}{2A_1\sqrt{\bar{h}_1 - \bar{h}_2}} \\ \frac{k_{12}}{2A_2\sqrt{\bar{h}_1 - \bar{h}_2}} & \frac{-1}{2A_2}\left(\frac{k_{12}}{\sqrt{\bar{h}_1 - \bar{h}_2}} + \frac{k_2}{\sqrt{\bar{h}_2}}\right) \end{bmatrix} = \begin{bmatrix} -0.1 & 0.1 \\ 0.05 & -0.1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1/A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1/A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Delta y = \Delta h_2 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix}$$

b) $G(s) = C\,[sI - A]^{-1} B$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+0.1 & -0.1 \\ -0.05 & s+0.1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix}}{(s+0.1)^2 - 0.005} \begin{bmatrix} \cancel{s+0.1} & \cancel{0.1} \\ 0.05 & s+0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{0.1}{s^2 + 0.2s + 0.005}$$

c) Doable if system observable

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.05 & -0.1 \end{bmatrix}$$

$\det \mathcal{O} \neq 0 \Rightarrow$ full rank $\Rightarrow$ observable

d) Controller can be made more aggressive, allowing larger control activity by

* reduce q_u

Making the level in tank 1 harder controlled relative to the level in tank 2. This can be achieved by

* increasing q_{11} in $Q_x = \begin{bmatrix} q_{11} & 0 \\ 0 & 1 \end{bmatrix}$

e) To get stationary accuracy when having low frequent disturbances, one should (try) add integral action and/or model the disturbance q_0 and estimate disturbance states

one answer good enough

Integral action

Introduce the integral state:

$$x_I = \int^t r - y \, dt \Rightarrow \dot{x}_I = r - y \approx r - \Delta h_2$$

add the state to the state space model

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \\ \dot{x}_I \end{bmatrix} = \underbrace{\begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.05 & -0.1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta x_I \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{B_e} \Delta u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Modify the cost function to (for example)

$$J = E\left\{ [\Delta h_1 \; \Delta h_2 \; x_I] \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q_I \end{bmatrix} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ x_I \end{bmatrix} + q_u \Delta u^2 \right\}$$

Minimizing J gives

$$u = -L_e \hat{x}_e$$

where $\hat{x}_e = [\hat{x} \; x_I]$, $\hat{x}$ being the estimated levels using a Kalman filter. The integral state should not be estimated!

The feedback gain is given by the stationary Ricatti eqns ($S > 0$)

$$A_e^T S + S A_e + Q_x - S B_e q_u^{-1} B_e^T S = 0$$

$$L_e = \frac{1}{q_u} B_e^T S = [L \; L_I]$$

while the estimator is given by

$$\dot{\hat{x}} = A\hat{x} + B\Delta u + K(\Delta y - C\Delta x)$$

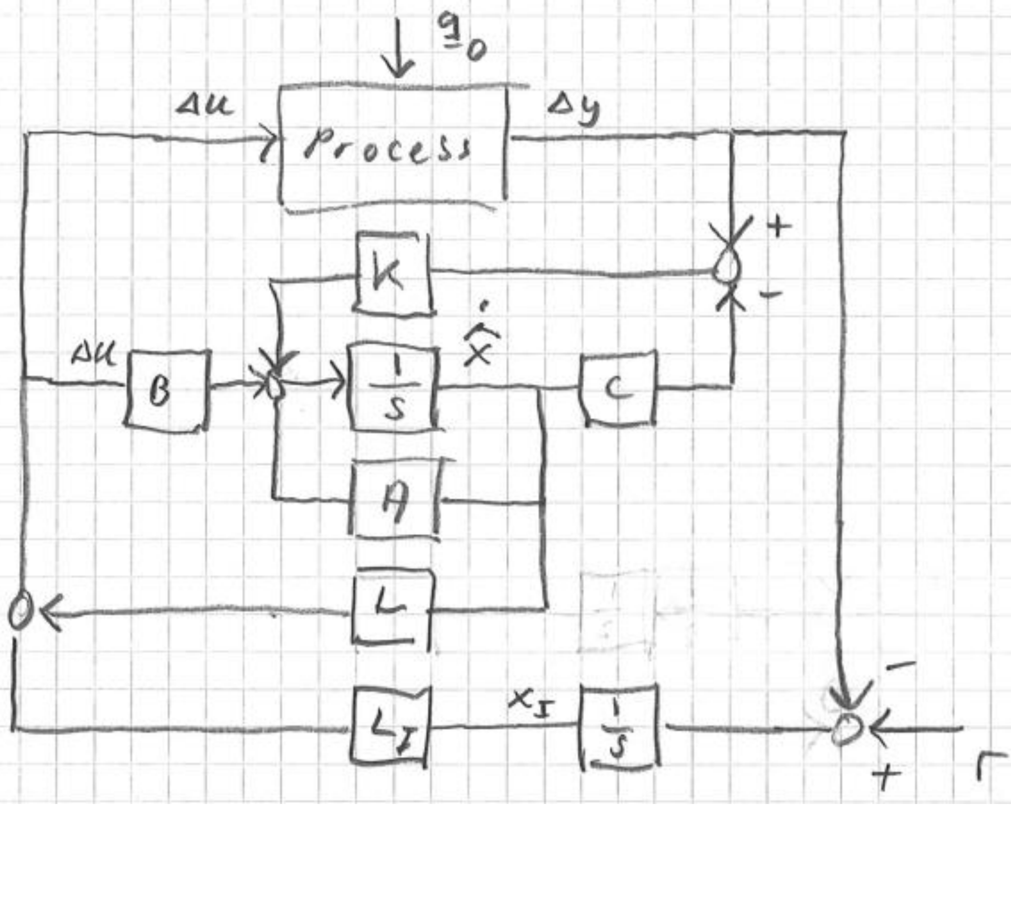
where K is the solution to $(P > 0)$

$$K = PC^T R_2^{-1}$$

$$AP + PA^T - PC^T R_2^{-1} CP + NR_1 N^T = 0$$

where R_1 is the intensity of the process disturbance and R_2 is the intensity of the measurement noise. Those are assumed independent here.

Blockscheme:



Adding disturbance model

Assume we have identified disturbance model such that

$$\Delta q_{T_0} = G_v(s) v_1$$

where v is assumed to be WGN

This can then be written on state space form

$$\dot{x}_v = A_v x_v + B_v v_1$$
$$\Delta q_{T_0} = C_v x_v$$

The original state space model is then extended with these states to get

$$\begin{bmatrix} \Delta \dot{h}_1 \\ \Delta \dot{h}_2 \\ \dot{x}_v \end{bmatrix} = \underbrace{\begin{bmatrix} -0.1 & 0.1 & C_v \\ 0.05 & -0.1 & 0 \\ 0 & 0 & A_v \end{bmatrix}}_{A_d} \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ x_v \end{bmatrix}}_{x_{ed}} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{B_d} \Delta u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ B_v \end{bmatrix}}_{N_d} v_1$$

$$\Delta y = \underbrace{[\,0 \quad 1 \quad 0\,]}_{C_d} x_{ed} + v_2$$

vector

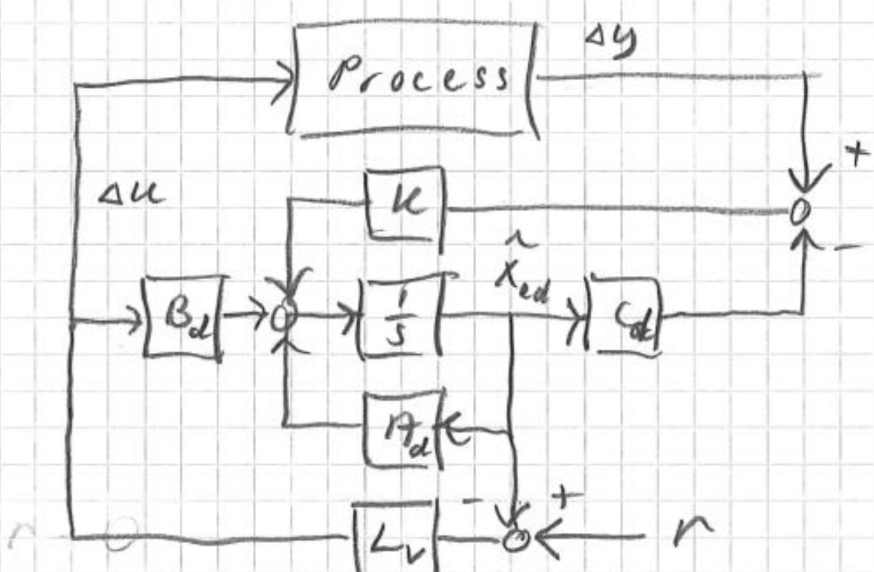
Now x_{ed} is estimated with a Kalman filter for (A_d, B_d and N_d) and the LQG feedback is calculated for the same model with zero cost on the disturbance states, i.e.

$$J = E\left\{[\Delta h_1\ \Delta h_2\ x_v] \underbrace{\begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{new } Q_x} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ x_v \end{bmatrix} + q_u \Delta u^2 \right\}$$

The stationary "Ricatti" eqns then give L_v such that the optimal feedback is

$$\Delta u = -L_v \begin{bmatrix} \Delta\hat{h}_1 \\ \Delta\hat{h}_2 \\ \hat{x}_v \end{bmatrix}$$

The block scheme is the std for LQG



2.

a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{e^{-Ts}}{1+s}$$

$$\Rightarrow \quad \dot{y}(t) + y(t) = u(t-T)$$

$$\begin{cases} \dot{x}(t) = -x(t) + u(t-T) \\ y(t) = x(t) \end{cases}$$

Sampling with $h = T$ gives

$$x(k+1) = \varphi x(k) + \gamma u(k-1)$$

where $\varphi = e^{-T}$

$$\gamma = \int_0^T e^{-\tau} d\tau = \left[-e^{-\tau} \right]_0^T = 1 - e^{-T}$$

Introduce $x_2(k) = u(k-1)$, which gives

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}}_{\tilde{x}(k+1)} = \underbrace{\begin{bmatrix} e^{-T} & 1-e^{-T} \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}}_{\tilde{x}(k)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \tilde{x}(k)$$

b) $u(k) = -L\tilde{x}(k)$

$$= -\begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Closed loop system

$$\tilde{x}(k+1) = (A - BL)\tilde{x}(k)$$

Poles equals the eigenvalues of $A - BL$

$$P(\lambda) = \det(\lambda I - A + BL)$$

$$\equiv (\lambda - 0.25)^2$$

$$= \lambda^2 - 0.5\lambda + 0.0625$$

$$T = \ln 2 \Rightarrow A = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}$$

$$P(\lambda) = \det \begin{bmatrix} \lambda - 0.5 & -0.5 \\ l_1 & \lambda + l_2 \end{bmatrix}$$

$$= \lambda^2 + \lambda(l_2 - 0.5) + 0.5(l_1 - l_2)$$

Polynomial identification

$$l_2 - 0.5 = -0.5 \Rightarrow l_2 = 0$$

$$0.5(l_1 - l_2) = 0.0625 \Rightarrow l_1 = l_2 + 0.125$$

$$= 0.125$$

$\therefore$ $u_{FB}(k) = -0.125\, x_1(k)$

$$= -0.125\,(0.5\, x_1(k-1) + 0.5\, x_2(k-1))$$

$$= -0.0625\, y(k-1) - 0.0625\, u_{FB}(k-1)$$

$$\Rightarrow u_{FB}(k) = \underbrace{\frac{-0.0625\, q^{-1}}{1 + 0.0625\, q^{-1}}}_{H(q)}\, y(k)$$

c) $u(k) = K_r r(k) - H(q) y(k)$

$y(k) = x_1(k)$

$= 0.5 x_1(k-1) + 0.5 u(k-2)$

System is stable $\Rightarrow$ we can assume steady-state

$$\begin{cases} \bar{u} = K_r \bar{r} - H(1) \bar{y} \\ \bar{y} = 0.5 \bar{y} + 0.5 \bar{u} \end{cases} \Rightarrow \bar{y} = \bar{u}$$

$\Rightarrow \bar{y} = K_r \bar{r} - H(1) \bar{y}$

$\bar{y} = \frac{K_r}{1 + H(1)} \bar{r} \equiv \bar{r}$ (correct gain)

$K_r = 1 + H(1) = 1 - \frac{0.0625}{1+0.0625} = \underline{\underline{0.9412}}$

3.

$$\dot{x} = \underbrace{\begin{bmatrix} -0.5 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{D} u$$

Observability matrix

$$A^2 = \begin{bmatrix} 0.25 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -0.5 & 1 & 1 \\ 0 & -1 & -1 \\ 0.25 & -1.5 & -1.5 \\ 0 & 1 & 1 \end{bmatrix}}$$

2nd & 3rd column equal

Rank = 2 $\Rightarrow$ Not observable

Is the system stable?

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+0.5 & -1 & -1 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

$$= (\lambda+0.5)(\lambda+1)(\lambda+1) = 0$$

All eigenvalues in LHP $\Rightarrow$ stable

$\Rightarrow$ Unobservable states must be stable

$\Rightarrow$ Detectable

Controllability matrix

$$\mathcal{S} = [B \quad AB \quad A^2B]$$

$$= \begin{bmatrix} 0 & 1 & 1 & -0.5 & \times & \times \\ 1 & 0 & -1 & 0 & \times & \times \\ 0 & 1 & 0 & -1 & \times & \times \end{bmatrix}$$

Linearly independent $\Rightarrow$ rank $= 3$

$\therefore$ Controllable

$\Rightarrow$ Stabilizable

4.

a) All step responses looks like step responses of 1st order systems with no time delay, i.e.

$$G_{ij}(s) = \frac{K_{ij}}{1+sT_{ij}} \quad , \; i,j = 1,2$$

The gains K_{ij} are the final value (since step size is 1) and the time constants are the times when appr. 63 % of the change has occurred.

This gives

$$G_{11}(s) \approx \frac{5}{1+s} \qquad G_{12} \approx \frac{2}{1+s}$$

$$G_{21}(s) \approx \frac{7.5}{1+0.5s} = \frac{15}{2+s}$$

$$G_{22}(s) \approx \frac{0.5}{1+0.5s} = \frac{1}{2+s}$$

$$\therefore \Rightarrow G(s) = \begin{bmatrix} \frac{5}{s+1} & \frac{2}{s+1} \\ \frac{15}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 5(s+2) & 2(s+2) \\ 15(s+1) & (s+1) \end{bmatrix}$$

b) $RGA(G(j\omega)) = G(j\omega) .* G^{-T}(j\omega)$

$$G^{-1}(s) = \frac{1}{25} \begin{bmatrix} -(s+1) & 2(s+2) \\ 15(s+1) & -5(s+2) \end{bmatrix}$$

$$G(s) .* G^{-T}(s) = \begin{bmatrix} \frac{-5}{25} & \frac{30}{25} \\ \frac{30}{25} & \frac{-5}{25} \end{bmatrix} = \begin{bmatrix} -0.2 & 1.2 \\ 1.2 & -0.2 \end{bmatrix}$$

Thus RGA does not depend on ω in this case.

Negative diagonal elements $\Rightarrow$ we should avoid diagonal pairing

Off diagonal elements close to 1

$\Rightarrow$ Choose pairing $y_1 \leftrightarrow u_2$
$y_2 \leftrightarrow u_1$

Alt. from plot only

From plot we see that

$$G(0) = \begin{bmatrix} 5 & 2 \\ 7.5 & 0.5 \end{bmatrix}$$

$$RGA(G(0)) = G(0) .* G^{-T}(0)$$

$$= \begin{bmatrix} 5 & 2 \\ 7.5 & 0.5 \end{bmatrix} .* \frac{1}{2.5-15} \begin{bmatrix} 0.5 & -7.5 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 & 1.2 \\ 1.2 & -0.2 \end{bmatrix}$$

Avoid diagonal coupling $\Rightarrow$ only option is

$y_1 \leftrightarrow u_2$
$y_2 \leftrightarrow u_1$

5.

$$\dot{x} = -x + u + w$$

$$\Phi_w = \frac{1}{1+\omega^2}$$

Spectral factorization gives that we can construct disturbance w as

$$v_1 \longrightarrow \boxed{G} \longrightarrow w$$

where $v_1 \sim WN(0,1)$ and $G(j\omega)G(-j\omega) = \Phi_w(\omega)$

$$\Phi_w(\omega) = \frac{1}{(1+j\omega)}\frac{1}{(1-j\omega)}$$

$$\Rightarrow \quad w(t) = \frac{1}{1+p} v_1(t)$$

$$(1+p)w = w + \dot{w} = v_1$$

Extended state space model:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ w \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_e} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{N_e} v_1$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_e} \begin{bmatrix} x \\ w \end{bmatrix} + v$$

where $v_2 \sim WN(0,1)$ and v_1 & v_2 are independent ($\Rightarrow R_1 = 1$, $R_2 = 1$, $R_{12} = 0$)

The optimal observer (minimizing the variance of the estimation error) is the continuous time Kalman filter, i.e.

$$\dot{\hat{x}}_e = A_e x_e + B_e u + K(y - C_e \hat{x}_e)$$

where K is the solution to (Theorem 5.4)

$$K = P C_e^T R_2^{-1} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{1} = \underline{\underline{\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}}}$$

$$AP + PA^T - PC_e^T R_2^{-1} (PC_e^T)^T + N_e R_1 N_e^T = 0$$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} [p_{11} \ p_{12}]$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ 1] = 0$$

$$\left.\begin{aligned} (1,1):&\quad (-p_{11} + p_{12})2 - p_{11}^2 = 0 \\ (1,2):&\quad -p_{12} + p_{22} - p_{12} - p_{11}p_{12} = 0 \\ (2,2):&\quad -p_{22} \cdot 2 - p_{12}^2 + 1 = 0 \end{aligned}\right\} \Rightarrow P$$

choose solution where $P > 0$

This eqn system is too hard to solve by hand. However, it was stated that

$$p_{11} \equiv Var\{\hat{x} - x\} = 0.2$$

$$(1,1) \Rightarrow \quad p_{12} = p_{11} + \frac{1}{2} p_{11}^2 = 0.2 + \frac{0.04}{2} = 0.22$$

$$\Rightarrow \quad K = \begin{bmatrix} 0.2 \\ 0.22 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.2 \\ 0.22 \end{bmatrix} \left(y - [1 \ 0] \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1.2 & 1 \\ -0.22 & -1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.2 \\ 0.22 \end{bmatrix} y$$

w scalar and can be modelled using spectral factorization means that w can be modelled as the output of a stable transfer fcn $G(s) = \frac{B(s)}{A(s)}$

Written on observer canonical form this transfer fcn can be realized as

$$\begin{bmatrix} \dot{w} \\ \dot{x}_w \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & 1 & 0 & \cdots \\ -a_2 & 0 & 1 & \ddots \\ \vdots & & & \\ \vdots & & & \end{bmatrix}}_{A_w} \begin{bmatrix} w \\ x_w \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \end{bmatrix}}_{B_w} v_1 \quad , \; v_1 \sim WN$$

Adding this to the original model

$$\begin{bmatrix} \dot{x} \\ \dot{w} \\ \dot{x}_w \end{bmatrix} = \underbrace{\begin{bmatrix} A & N & 0 & \cdots & 0 \\ 0 & [& A_w & &] \end{bmatrix}}_{A_e} \begin{bmatrix} x \\ w \\ x_w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \underbrace{\begin{bmatrix} 0 \\ B_w \end{bmatrix}}_{N_e} v_1$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_e} \begin{bmatrix} x \\ w \\ x_w \end{bmatrix} + v_2$$

Requirements for $\exists$ Kalman filter are stated in Lemma 5.1. For this problem they are

(i) $R_1 \geq 0$ (Obviously fullfilled)

(ii) (A_e, C_e) detectable
(iii) (A_e, R_1) stabilizable

$\Big\}$ Fullfilled if A_e is stable

A stable, A_w stable $\det(\lambda I - A_e) =$
$= \det(\lambda I - A)\det(\lambda I - A_w) \Rightarrow A_e$ stable

$\therefore$ Yes, one can always determine a KF!