CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Wednesday April 3, 2024

Time and place:	08:30 - 12:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

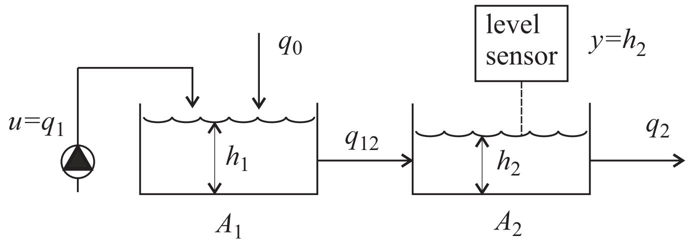
3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

Solutions to problems and exercises are not allowed in the notes!

Mobile telephones, laptops or tablets/iPads are not allowed!

1. Consider the system below with two tanks in series. To the first tank there is an uncontrollable flow q_0 of a fluid. In order to control the levels in the tanks a controller is to be installed using a level sensor, sensing the level in the second tank, and a pump delivering a flow q_1 to the first tank.



According to Bernoulli's law the flow between the two tanks is given by

$$q_{12} = k_{12}\sqrt{h_1 - h_2}$$

and the flow leaving the second tank is $q_2 = k_2\sqrt{h_2}$.

The bottom areas of the two tanks are $A_1 = 0.5\ \mathrm{m}^2$ and $A_2 = 1\ \mathrm{m}^2$, and $k_{12} = k_2 = 0.1\ \mathrm{m}^{5/2}\mathrm{s}^{-1}$.

Note that all questions can be answered independantly of eachother!

(a) Linearize the system and show that around the operating point $\bar{h}_1 = 2$ m and $\bar{h}_2 = 1$ m the response from pumpflow $u = q_1$ to the measured level $y = h_2$ is given by the state space model

$$\begin{aligned} \frac{d}{dt}\Delta h(t) &= \begin{bmatrix} -0.1 & 0.1 \\ 0.05 & -0.1 \end{bmatrix} \Delta h(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Delta u(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Delta q_0(t) \\ \Delta y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \Delta h(t) \end{aligned}$$

where Δ denotes deviations from stationary values in the operating point.

3 p.

(b) Determine the corresponding transfer function from Δu to Δy.

1 p.

(c) Even though the main purpose is to keep the level h_2 constant (to give a constant flow q_2 to the next stage in the process) it is of course important to monitor the level in tank 1 as well. To save the money though, it would be preferable if we could estimate it instead. Is that doable?

2 p.

(d) It is decided that both levels should be controlled using a stationary LQG controller minimizing

$$J = E\{\Delta h^T Q_x \Delta h + q_u \Delta u^2\}$$

From the beginning the tuning matrices were chosen to be $Q_x = I$ and q_u=1, but then it turned out that the first tank sometimes was flooded. What two changes would you consider first when retuning?

1 p.

(e) Another issue was found to be that the disturbances on q_0 could be low frequent, causing the level y to be off setpoint for long periods. What addition to your controller would you suggest?

Please state the matrices and equations to solve (you need not solve them!), and draw a block diagram showing the complete control structure.

3 p.

2. A continuous time linear system is given by

$$y(t) = \begin{bmatrix} 0 & \frac{2}{p+1} \\ 0 & \frac{1}{p} \\ \frac{1}{p+1} & \frac{2}{p+1} \end{bmatrix} u(t)$$

Note: p is the time derivative operator (d/dt) which can be used in calculations for transfer function operators, such as the elements in the above matrix, in the same way as the Laplace variable s is used for transfer functions.

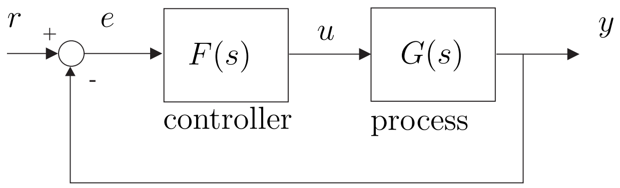
(a) What are the poles and zeros of this multivariable system?

2 p.

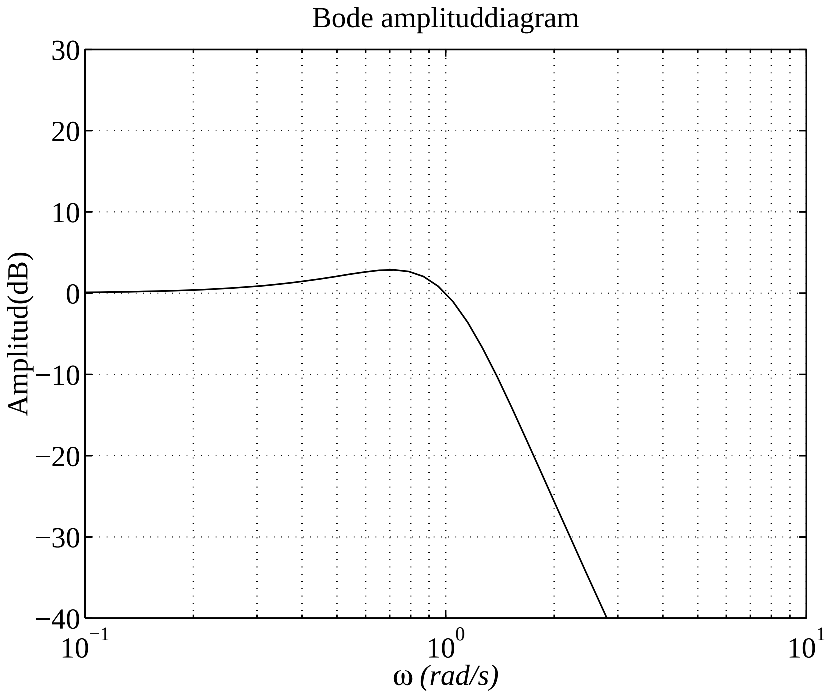
(b) Give a state space realization of the system. Is it minimal?

2 p.

3. A controller $F(s)$ has been determined for a process model with transfer function $G(s)$ (see figure).



The Bode diagram for the corresponding transfer function from r to y is given below.



In the modelling some dynamics and uncertainties were ignored. The 'true' transfer function for the process is

$$G_0(s) = \frac{K}{1 + 0.5s} G(s), \qquad 0.9 < K < 1.1$$

where we have included an uncertainty of the stationary gain K.

Can we trust that the feedback system will remain stable in spite of the simplifications made in the design of the controller $F(s)$?

If you sketch in the Bode diagram above please do not forget to include that when you submit your answers.

3 p.

4. Consider the system

$$G(s) = \frac{1}{s}e^{-0.6s}$$

from input $u(t)$ to output $y(t)$.

This system is to be discretized for the sampling time $h = 1$ and zero order hold input. The difficulty here is that there is a time delay that is not equal to a multiple of the sampling time.

(a) Express the above model as a continuous time LTI state space model with delayed input.

1 p.

(b) Use the analytical solution to the state space model to derive the corresponding discrete time state space model on standard form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

for which $y(k) = y(kh) = y(t)$ if $u(k) = u(t)$ for all $t = kh$.

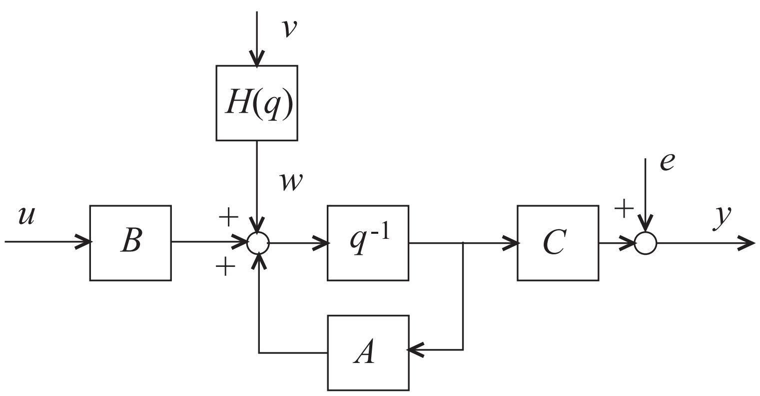
2 p.

5. Assume we have a system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + e(k) \end{aligned}$$

where e is a measurement white gaussian noise (WGN).

The process disturbance w is not white, though it originates from a source v that can be considered as WGN, see the figure below.



Neither w nor v can be measured online to be used in a feed forward control. However, from separate system identification experiments the pulse transfer operator H from v to w has been determined to be

$$H = \frac{\beta q^{-1}}{1 + \alpha_1 q^{-1} + \alpha_2 q^{-2}}$$

This can be used in an LQG controller by extending the state-space model and then estimate and feed back all states, i.e. the original process states and the disturbance states.

(a) Determine a state space model

$$\begin{aligned} x_w(k+1) &= A_w x_w(k) + B_w v(k) \\ w(k) &= C_w x_w(k) \end{aligned}$$

corresponding to H.

2 p.

(b) Give an extended state space model, including the disturbance states, on the correct form for design of a Kalman filter (you need not calculate the filter).

2 p.

6. When describing the input-output behaviour of a system, uncontrollable and unobservable states (modes) can be removed. Another way to (further) reduce the number of states is to look at the time constants related to different states (modes). If some states have very fast dynamics relative to the others they can often be assumed to be in a steady state, which can be used to reduce the model order.

   The state-space equations for a system has been determined by physical modelling:

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \underbrace{\begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -0.1 & 0 & 0 \\ 0.5 & -1 & -1 & -10 \\ 0 & 1 & 7 & -100 \end{bmatrix}}_{A} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)
\end{aligned}
$$

   Running the command `[V,D]=eig(A)` in Matlab gives

$$
\begin{aligned}
V &= \begin{bmatrix} -0.8704 & 0.7675 & 0.0113 & -0.3353 \\ 0 & 0 & 0 & 0.7237 \\ 0.4911 & 0.6395 & -0.1013 & -0.6021 \\ 0.0353 & 0.0453 & -0.9948 & -0.0349 \end{bmatrix} \\
D &= \begin{bmatrix} -2.6048 & 0 & 0 & 0 \\ 0 & -1.1078 & 0 & 0 \\ 0 & 0 & -99.2874 & 0 \\ 0 & 0 & 0 & -0.1000 \end{bmatrix}
\end{aligned}
$$

   where the columns of V are the eigenvectors of A, and the elements on the diagonal of D are the eigenvalues. Then taking the inverse of V gives

$$
V^{-1} = \begin{bmatrix} -0.6851 & 0.3676 & 0.8287 & -0.0921 \\ 0.5261 & 1.0204 & 0.9387 & -0.0896 \\ -0.0004 & 0.0109 & 0.0721 & -1.0126 \\ 0 & 1.3817 & 0 & 0 \end{bmatrix}
$$

   (a) Use the above information to diagonalize the state-space equation.

   3 p.

   (b) Suggest, based on an analysis of the diagonalized system, a reduced state-space model that should give about the same input-output behaviour as the original model.

   3 p.

Good luck!