7) Volume balances for the two tanks give

$$\frac{d}{dt}(A_1 h_1) = q_1 + q_0 - q_{12}$$

$$\frac{d}{dt}(A_2 h_2) = q_{12} - q_2$$

which gives the nonlinear state equations

$$\dot{h}_1 = \frac{1}{A_1} u + \frac{1}{A_1} q_0 - \frac{k_{12}}{A_1}\sqrt{h_1 - h_2} = f_1(h, u, q_0)$$

$$\dot{h}_2 = \frac{k_{12}}{A_2}\sqrt{h_1 - h_2} - k_2\sqrt{h_2} = f_2(\cdot\ \cdot)$$

a) Linearization around $\bar{h}_1 = 2$ and $\bar{h}_1 = 1$

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{df_1}{dh_1} & \frac{df_1}{dh_2} \\ \frac{df_2}{dh_1} & \frac{df_2}{dh_2} \end{bmatrix}_{(\bar{h},\bar{u})}}_{A} \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{df_1}{du} \\ \frac{df_2}{du} \end{bmatrix}}_{B} \Delta u + \underbrace{\begin{bmatrix} \frac{df_1}{dq_0} \\ \frac{df_2}{dq_0} \end{bmatrix}}_{N} \Delta q_0$$

$$A = \begin{bmatrix} \frac{-k_{12}}{2A_1\sqrt{\bar{h}_1 - \bar{h}_2}} & \frac{k_{12}}{2A_1\sqrt{\bar{h}_1 - \bar{h}_2}} \\ \frac{k_{12}}{2A_2\sqrt{\bar{h}_1 - \bar{h}_2}} & \frac{-1}{2A_2}\left(\frac{k_{12}}{\sqrt{\bar{h}_1 - \bar{h}_2}} + \frac{k_2}{\sqrt{\bar{h}_2}}\right) \end{bmatrix} = \begin{bmatrix} -0.1 & 0.1 \\ 0.05 & -0.1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1/A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1/A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Delta y = \Delta h_2 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix}$$

b) $G(s) = C\,[sI - A]^{-1} B$

$$= [0 \;\; 1] \begin{bmatrix} s+0.1 & -0.1 \\ -0.05 & s+0.1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{[0 \;\; 1]}{(s+0.1)^2 - 0.005} \begin{bmatrix} s+0.1 & 0.1 \\ 0.05 & s+0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{0.1}{s^2 + 0.2s + 0.005}$$

c) Doable if system observable

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.05 & -0.1 \end{bmatrix}$$

$\det \mathcal{O} \neq 0 \Rightarrow$ full rank $\Rightarrow$ observable

d) Controller can be made more aggressive, allowing larger control activity by

* reduce q_u

Making the level in tank 1 harder controlled relative to the level in tank 2. This can be achieved by

* increasing q_{11} in $Q_x = \begin{bmatrix} q_{11} & 0 \\ 0 & 1 \end{bmatrix}$

e) To get stationary accuracy when having low frequent disturbances, one should (try) add integral action and/or model the disturbance q_0 and estimate disturbance states

one answer good enough

Integral action

Introduce the integral state:

$$x_I = \int^t r - y \, dt \Rightarrow \dot{x}_I = r - y \approx r - \Delta h_2$$

add the state to the state space model

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \\ \dot{x}_I \end{bmatrix} = \underbrace{\begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.05 & -0.1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta x_I \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{B_e} \Delta u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

Modify the cost function to (for example)

$$J = E\left\{ [\Delta h_1 \; \Delta h_2 \; x_I] \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q_I \end{bmatrix} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ x_I \end{bmatrix} + q_u \Delta u^2 \right\}$$

Minimizing J gives

$$u = -L_e \hat{x}_e$$

where $\hat{x}_e = [\hat{x} \; x_I]$, $\hat{x}$ being the estimated levels using a Kalman filter. The integral state should not be estimated!

The feedback gain is given by the stationary Ricatti eqns ($S > 0$)

$$A_e^T S + S A_e + Q_x - S B_e q_u^{-1} B_e^T S = 0$$

$$L_e = \frac{1}{q_u} B_e^T S = [L \; L_I]$$

while the estimator is given by

$$\dot{\hat{x}} = A\hat{x} + B\Delta u + K(\Delta y - C\Delta x)$$

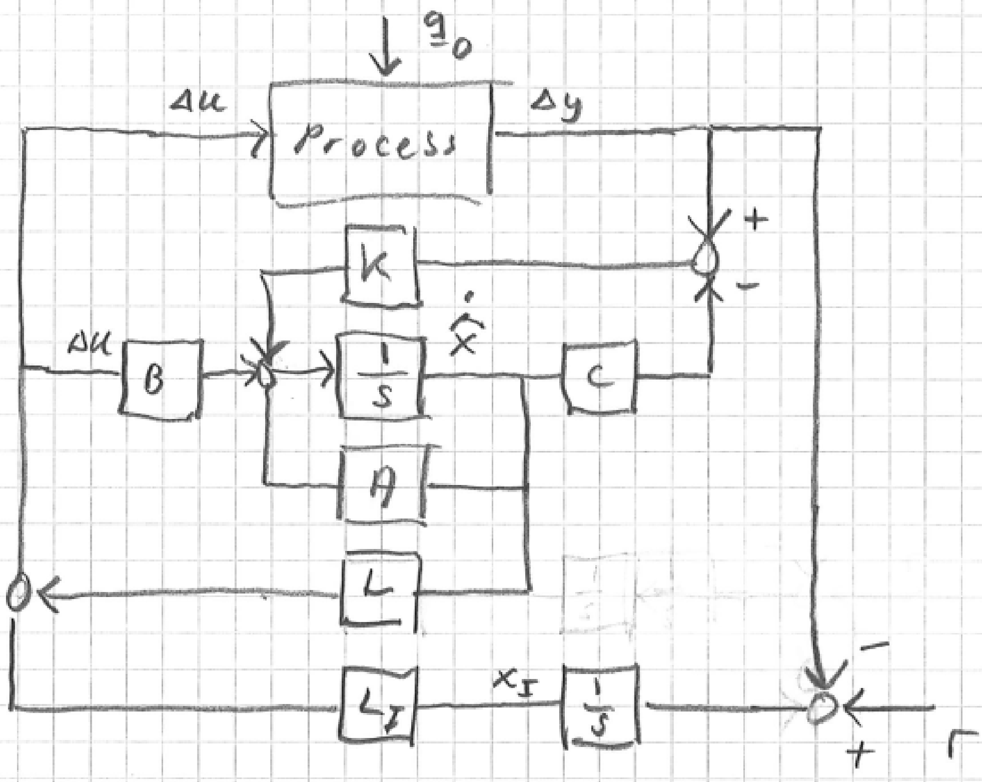
where K is the solution to $(P > 0)$

$$K = P C^T R_2^{-1}$$

$$AP + PA^T - P C^T R_2^{-1} C P + N R_1 N^T = 0$$

where R_1 is the intensity of the process disturbance and R_2 is the intensity of the measurement noise. Those are assumed independent here.

Blockscheme:



2)

$$G(p) = \begin{bmatrix} 0 & \frac{2}{p+1} \\ 0 & \frac{1}{p} \\ \frac{1}{p+1} & \frac{2}{p+1} \end{bmatrix}$$

a) Largest minors (order 2)

$$0 \,,\; -\frac{1}{p(p+1)} \,,\; \frac{-2}{(p+1)^2}$$

Order 1 = elements of G

The polepolynomial is the least common divisor of all minors, i.e.

$$P(p) = p(p+1)^2$$

$\therefore$ 3 poles: $0, -1, -1$

The minors are consequently

$$\frac{0}{P(p)} \,,\; \frac{p+1}{P(p)} \,,\; \frac{2p}{P(p)}$$

Zero polynomial is the largest common divisor, which is 1 in this case

$\Rightarrow$ No zeros

b) $y_1 = \frac{2}{p+1} u_2$

$y_2 = \frac{1}{p} u_2$

$y_3 = \frac{1}{p+1} u_1 + \frac{2}{p+1} u_2$

Let, <u>for example</u>

$x_1 = \frac{1}{p+1} u_2, \quad x_2 = \frac{1}{p} u_2, \quad x_3 = \frac{1}{p+1} u_1$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Minimal since 3 states and 3 poles

(observable and controllable equivalently)

3) Robust stability (p 152-155)

$$G_0 = (1+\Delta_G)G = \frac{K}{1+0.5s} G(s)$$

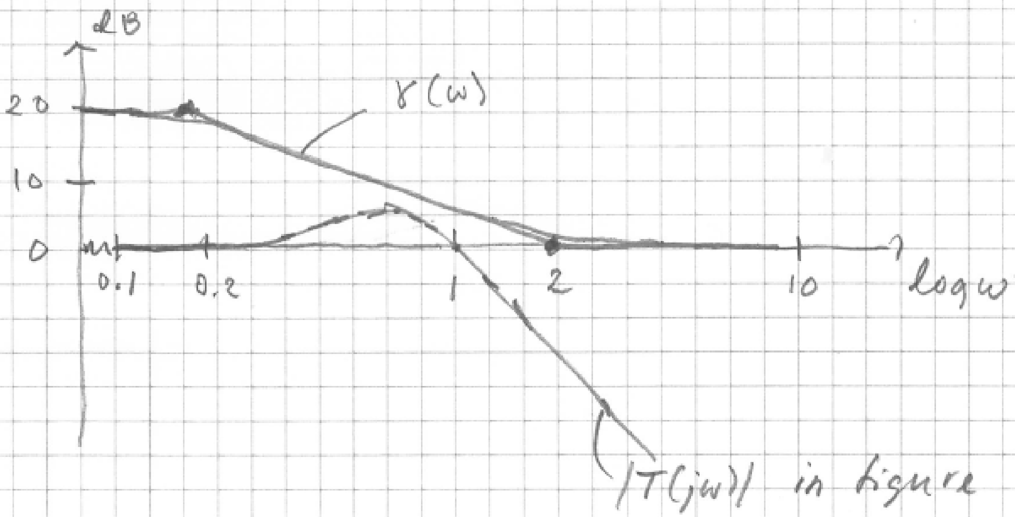
$$\Rightarrow \Delta_G = \frac{K}{1+0.5s} - 1 = \frac{K-1-0.5s}{1+0.5s}, \quad 0.9 < K < 1.1$$

System is stable if $\|\Delta_G T\|_\infty < 1$, i.e.

$$|T(j\omega)| < \frac{1}{|\Delta_G(j\omega)|} = \frac{|1+0.5j\omega|}{|K-1-0.5j\omega|}$$

$$\geq \underbrace{\frac{|1+0.5j\omega|}{|1-5j\omega|} \cdot 10}_{\gamma(\omega)}$$

So if $|T(j\omega)|$, which is plotted in the figure is lower than $\gamma(\omega)$ the system remains stable. $\gamma(\omega)$ is on Bode form and has the amplitude diagram



∴ System is stable!

4)

$$G(s) = \frac{1}{s} e^{-0.6s}$$

Time delay $T_d = 0.6$, Sampling time $h = 1$

We recognise this process as a pure integration ($1/s$) with a time delay, i.e.

a) $\dot{x}(t) = a x(t) + b u^*(t)$, $u^*(t) = u(t-0.6)$

$y(t) = c x(t)$

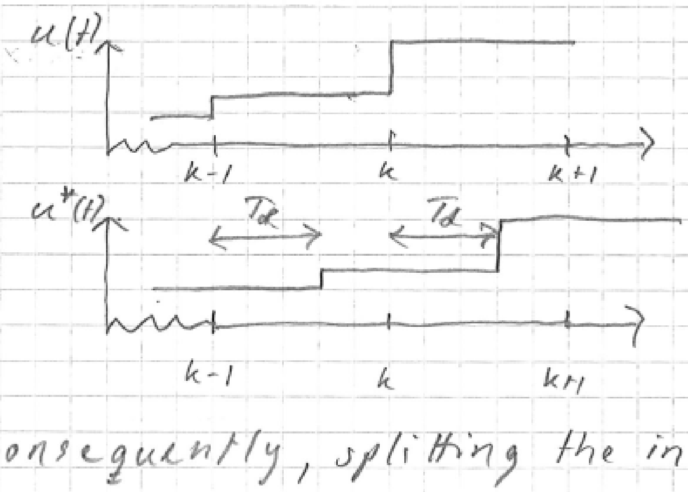
with $a = 0$, $b = 1$, $c = 1$

b) Given the state x_0 at time t_0 the analytical solution is

$$x(t) = e^{a(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} B u^*(\tau) d\tau$$

Let $t_0 = kh = k$ and $t = (k+1)h = k+1$.

Study input signal on the interval



Consequently, splitting the interval into two

$$x(k+1) = x(k) + \int_{k}^{k+T_d} B u(k-1) d\tau + \int_{k+T_d}^{k+1} B u(k) d\tau$$

$$x(k+1) = x(k) + T_d u(k-1) + (1-T_d)u(k)$$

Introduce delayed input state

$$x_d(k) = u(k-1)$$

Then with $T_d = 0.6$

$$\begin{bmatrix} x(k+1) \\ x_d(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_d(k) \end{bmatrix} + \begin{bmatrix} 0.4 \\ 1 \end{bmatrix} u(k)$$

5)

a)

$$w(k) = \frac{\beta q^{-1}}{1 + \alpha_1 q^{-1} + \alpha_2 q^{-2}} v(k)$$

$$w(k) + \alpha_1 w(k-1) + \alpha_2 w(k-2) = \beta v(k-1)$$

$$w(k+1) + \alpha_1 w(k) + \alpha_2 w(k-1) = \beta v(k)$$

$$\underbrace{\begin{bmatrix} w(k+1) \\ w(k) \end{bmatrix}}_{x_w(k+1)} = \underbrace{\begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix}}_{A_w} \underbrace{\begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}}_{x_w(k)} + \underbrace{\begin{bmatrix} \beta \\ 0 \end{bmatrix}}_{B_w} v(k)$$

$$w(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_w} \begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}$$

b)

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ w(k+1) \\ w(k) \end{bmatrix}}_{x_e(k+1)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -\alpha_1 & -\alpha_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ w(k) \\ w(k-1) \end{bmatrix}}_{x_e(k)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ \beta \\ 0 \end{bmatrix} v(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_e(k) + e(k)$$

6)

a) Definition of eigenvector v_i and eigenvalue λ_i

$$A v_i = \lambda_i v_i$$

$$\Rightarrow \quad AV = VD, \quad V = [v_1 \; v_2 \dots v_n], \; D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Let $z = V^{-1} x \iff x = Vz$

$$\dot{z} = V^{-1}\dot{x} = V^{-1}(Ax + Bu)$$

$$= \underbrace{V^{-1} A V}_{D} z + V^{-1} B u$$

$$= \underbrace{\begin{bmatrix} -2.6 & & & 0 \\ & -1.11 & & \\ & & -99.3 & \\ 0 & & & -0.1 \end{bmatrix}}_{A_z} z + \underbrace{\begin{bmatrix} -0.27 \\ 1 \\ 0.036 \\ 0 \end{bmatrix}}_{B_z} u$$

$$y = Cx = CVz$$

$$= \underbrace{\begin{bmatrix} -0.38 & 1.4 & -0.09 & -0.94 \\ 0.035 & 0.045 & -0.99 & 0.69 \end{bmatrix}}_{C_z} z$$

b) The time constants for the diagonalized states relate to the eigenvalues as

$$T_i = -\frac{1}{\lambda_i} \quad \Rightarrow \quad T = (0.4, 0.9, 0.01, 10)$$

We see that z_3 has much faster dynamics than the others

If we set z_3 to be in equilibrium (steady-state) we have

$$0 = -99.3\, z_3 + 0.036\, u$$

$$\Rightarrow z_3 = 0.00036\, u$$

Since z_3 does not affect the other states the corresponding rows and column can be removed from A_z and B_z.

The output equation, however, need to be updated:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -2.6 & 0 & 0 \\ 0 & -1.11 & 0 \\ 0 & 0 & -0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \end{bmatrix} + \begin{bmatrix} -0.27 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -0.38 & 1.4 & -0.94 \\ 0.035 & 0.045 & 0.69 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \end{bmatrix} + \underbrace{\begin{bmatrix} -0.09 \cdot 0.00036 \\ -0.99 \cdot 0.00036 \end{bmatrix}}_{\begin{bmatrix} 3.24 \cdot 10^{-5} \\ 3.58 \cdot 10^{-4} \end{bmatrix}} u$$

As can also be seen z_4 is not affected by u, it does not affect the other states and it is stable. Therefore $z_4 \to 0$ and can also be removed

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2.6 & 0 \\ 0 & -1.11 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -0.27 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -0.38 & 1.4 \\ 0.035 & 0.045 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 3.24 \cdot 10^{-5} \\ 3.58 \cdot 10^{-4} \end{bmatrix} u$$