LCSD 240108

1 a) $x(k+1) = x(k) + v_1(k)$, $v_1 \sim WGN(0, r_1 I)$

$y(k) = x(k) + v_2(k)$, $v_2 \sim WGN(0, r_2 I)$

A Kalman filter (filter case, 5.101) gives estimates with the lowest estimation error variance.

With $A = 1$, $B = 0$, $C = 1$, $R_{12} = 0$, $R_1 = r_1$, $R_2 = r_2$, $N = 1$

we get $\hat{x}(k+1|k) = \hat{x}(k|k) \equiv \hat{x}(k)$

$$\hat{x}(k) = \hat{x}(k-1) + \tilde{K}(y(k) - \hat{x}(k-1))$$

where $\tilde{K} = P(P + r_2)^{-1}$ and (5.99)

$$P \equiv P + r_1 - P(P + r_2)^{-1} P$$

$$0 = r_1(P + r_2) - P^2$$

$$= -\left(P - \frac{r_1}{2}\right)^2 + \frac{r_1^2}{4} + r_1 r_2$$

$$P = \frac{r_1}{2} \underset{(-)}{+} \sqrt{\frac{r_1^2}{4} + r_1 r_2} = \frac{1}{2}\left(r_1 + \sqrt{r_1^2 + 4 r_1 r_2}\right)$$

$$\Rightarrow \tilde{K} = P(P + r_2)^{-1}$$

$$= \frac{\frac{1}{2}\left(r_1 + \sqrt{r_1^2 + 4 r_1 r_2}\right)}{r_2 + \frac{1}{2}\left(r_1 + \sqrt{r_1^2 + 4 r_1 r_2}\right)}$$

$$= \frac{r_1 + \sqrt{r_1^2 + 4 r_1 r_2}}{2 r_2 + r_1 + \sqrt{r_1^2 + 4 r_1 r_2}}$$

$r_1 = 1$ and $r_2 = 2$

$$\Rightarrow P = \frac{1}{2}\left(1 + \sqrt{1 + 8}\right) = 2$$

$$\tilde{K} = \frac{2}{2+2} = 0.5$$

b) The error variance is given by

$$E\{\hat{x}(k)^2\} = (1-\tilde{K})P(1-\tilde{K}) + \tilde{K}\, r_2\, \tilde{K}$$
$$= 0.5^2 \cdot 2 + 0.5^2 \cdot 2 = \underline{\underline{1}}$$

∴ Variance has been halved compared to the measurement $y(k)$

2) For

$$v = WGN(0,1) \longrightarrow \boxed{G} \longrightarrow w \quad , \; G \text{ stable}$$

we have that

$$\Phi_w(\omega) = |G(j\omega)|^2 = G(j\omega)\,G(-j\omega)$$

Thus

$$\Phi(\omega) = \frac{1}{1+\omega^2} = \frac{1}{1+j\omega}\,\frac{1}{1-j\omega}$$

and consequently

$$G(s) = \frac{1}{1+s}$$

3 a) $Y_1 = \frac{1}{4+s} u \Rightarrow \dot{y}_1 + 4y_1 = u$

$Y = \frac{1}{1+s} y_1 \Rightarrow \dot{y} + y = y_1$

Let $x_1 = y$ and $x_2 = y_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $x_I = \int_0^t e(\tau)\, d\tau \Rightarrow \dot{x}_I = r - y = r - Cx$

$$= r - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Extended state space model

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_I \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{A_e} \begin{bmatrix} x_1 \\ x_2 \\ x_I \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{B_e} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_e} x_e$$

c) $J = \int_0^\infty y^2 + u^2 + q_I x_I^2 \, dt$

$$= \int_0^\infty \begin{bmatrix} x_1 & x_2 & x_I \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_I \end{bmatrix}}_{Q_x} \begin{bmatrix} x_1 \\ x_2 \\ x_I \end{bmatrix} + u \cdot \underbrace{1}_{Q_u} \cdot u \, dt$$

The controller that minimizes J is the LQR solution

$$u = -L_e x_e$$

where L_e is given by the solution to

$$L_e = Q_u^{-1} B_e^T S \qquad (Q_{xu} = [0 \ 0])$$

$$0 = A_e^T S + S A_e + Q_x - S B_e Q_u^{-1} (S B_e)^T$$

where $S_{3\times3}$ symmetric and $\succ 0$

d) From the block scheme we have

$$U(s) = K\left(-Y_1(s) - \left(K_p + \frac{K_I}{s}\right) Y(s)\right)$$

$$= -K Y_1(s) - K K_p Y(s) - \frac{K K_I}{s} Y(s)$$

which corresponds to

$$u(t) = -K x_2(t) - K K_p x_1(t) + K K_I x_I(t)$$

$$= -[l_1 \ l_2 \ l_3] x_e$$

$$\Rightarrow K = l_2, \quad K K_p = l_1 \Rightarrow K_p = \frac{l_1}{l_2}, \quad l_3 = K K_I \Rightarrow K_I = -\frac{l_3}{l_2}$$

e) Increase the integral action, i.e.

Increase q_I and recalculate L

4 a) $x_1 = S$, $x_2 = X$, $u = X_{in}$, $v = S_{in}$ and

$V = 1$, $Q = 2$, $\mu = 1$, $Y = 0.1$ and $K = 1$ gives

$$\begin{cases} \dot{x}_1 = 2v - 2x_1 - 10\dfrac{x_1 x_2}{1+x_1} = f_1(x,u,v) \\ \dot{x}_2 = 2u - 2x_2 + \dfrac{x_1 x_2}{1+x_1} = f_2(x,u,v) \end{cases}$$

b) Equilibrium and $\bar{v} = 6$, $\bar{x}_1 = 1$

$$\begin{cases} 0 = 12 - 2 - 10\dfrac{\bar{x}_2}{2} = 10 - 5\bar{x}_2 \\ 0 = 2\bar{u} - 2\bar{x}_2 + \dfrac{\bar{x}_2}{2} = 2\bar{u} - 1,5\bar{x}_2 \end{cases}$$

$\Rightarrow \bar{x}_2 = 2$, $\bar{u} = 1.5$

Let $\Delta x = x - \bar{x}$, $\Delta u = u - \bar{u}$ and $\Delta v = v - \bar{v}$.

Then the linearized model is given by

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}_{(\bar{x},\bar{u},\bar{v})} \begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix} + \begin{bmatrix}\partial f_1/\partial u \\ \partial f_2/\partial u\end{bmatrix}_{(\bar{x},\bar{u},\bar{v})} \Delta u + \begin{bmatrix}\partial f_1/\partial v \\ \partial f_2/\partial v\end{bmatrix}_{(\bar{x},\bar{u},\bar{v})} \Delta v$$

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{(\bar{x},\bar{u},\bar{v})} = -2 - 10\bar{x}_2 \frac{1}{(1+\bar{x}_1)^2} = -12$$

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{(\bar{x},\bar{u},\bar{v})} = -10\frac{\bar{x}_1}{1+\bar{x}_1} = -5$$

$$\left.\frac{df_2}{dx_1}\right|_{(\bar{x}_1,\bar{u},\bar{v})} = \bar{x}_2 \frac{1}{(1+\bar{x}_1)^2} = 0.5, \quad \frac{df_2}{dx_2} = -2 + \frac{\bar{x}_1}{1+\bar{x}_1} = -1.5$$

f_1 and f_2 are already linear w.r.t. u and v

Thus,

$$\Delta\dot{x} = \underbrace{\begin{bmatrix} -7 & -5 \\ 0.5 & -1.5 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B} \Delta u + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{N} \Delta v$$

c) If we measure the organics (x_1) then

$$\Delta y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}\Delta x$$

and the observability matrix is then

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -7 & -5 \end{bmatrix}$$

which has full rank since $\det(\mathcal{O}) = -5 \neq 0$

Thus, the system is observable and we can then estimate x_2.

d) A standard observer on innovation form:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \\ &= (A - KC)\hat{x} + Bu + Ky \end{aligned}$$

The poles of the observer are the eigenvalues of $A - KC$.

$$\det(\lambda I - A + KC) = \det\left\{\begin{bmatrix} \lambda+7 & 5 \\ -0.5 & \lambda+1.5 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right\}$$

$$= \det \begin{bmatrix} \lambda + 7 + k_1 & 5 \\ -0.5 + k_2 & \lambda + 1.5 \end{bmatrix} =$$

$$= \lambda^2 + \lambda(7 + k_1 + 1.5) + 2.5 - 5k_2 + 10.5 + 1.5k_1$$

If both eigenvalues are in $-p$ this should equal

$$p(\lambda) = (\lambda + p)^2 = \lambda^2 + 2p\lambda + p^2$$

Identification of coefficients

$$8.5 + k_1 = 2p \Rightarrow k_1 = 2p - 8.5$$

$$13 + 1.5k_1 - 5k_2 = p^2 \Rightarrow k_2 = \frac{1}{5}(0.25 + 3p - p^2)$$

$$p = 5 \Rightarrow K^T = [1.5 \quad -1.95]$$

e) In general, the faster the observer (p larger $\Leftrightarrow$ $\lambda_{1,2}$ further to the left i LHP) the more sensitive the estimate is to measurement noise.

increasing p $\Rightarrow$ $|K|$ larger (see above)

Including noise v_2 in measurement the estimation error $\tilde{x} = \hat{x} - x$ is given by

$$\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x} = A(x - \hat{x}) + K(Cx + v_2 - C\hat{x})$$

$$= (A - KC)\tilde{x} + Kv_2$$

As K gets large v_2 affects the estimate strongly. To reduce this, try reducing p

5 a) All step responses look like responses of 1st order systems with no time delay, i.e.

$$G_{ij} = \frac{K_{ij}}{1+sT_{ij}} \quad , \quad i,j = 1,2$$

The gains are the final value (since step size is 1) and the time constants are the times when appr 63% of the change has occurred.

This gives

$$G_{11} = \frac{5}{1+s} \quad , \quad G_{12} = \frac{-2}{1+s}$$

$$G_{21} = \frac{5}{1+0.5s} = \frac{10}{2+s}$$

$$G_{22} = \frac{0.5}{1+0.5s} = \frac{1}{2+s}$$

$$G(s) \approx \begin{bmatrix} \frac{5}{1+s} & \frac{-2}{1+s} \\ \frac{10}{2+s} & \frac{1}{2+s} \end{bmatrix} = \frac{1}{(1+s)(2+s)} \begin{bmatrix} 5(s+2) & -2(s+2) \\ 10(s+1) & (s+1) \end{bmatrix}$$

b) $RGA(j\omega) = G(j\omega).*(G^{-1}(j\omega))^T$

$$G^{-1}(s) = \frac{1}{25} \begin{bmatrix} s+1 & 2(s+2) \\ -10(s+1) & 5(s+2) \end{bmatrix}$$

$$RGA(G(s)) = \begin{bmatrix} 5/25 & 20/25 \\ 20/25 & 5/25 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

Thus RGA does not depend on ω in this case.

Off diagonal elements closer to 1 suggests

$u_1 - y_2$, $u_2 - y_1$

Reordering u to have diagonal pairing:

$$\tilde{G}(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} -2(s+2) & 5(s+2) \\ s+1 & 10(s+1) \end{bmatrix}$$

The Niederlinsky index

$$NI = \frac{\det(\tilde{G}(0))}{\prod \tilde{G}_{ii}(0)}$$

$$\tilde{G}(0) = \begin{bmatrix} -1 & 2.5 \\ 0.5 & 5 \end{bmatrix} \Rightarrow NI = \frac{-6.25}{-5} = 1.25 > 0 \quad \underline{\underline{\text{ok!}}}$$

No negative elements in $RGA(G(0))$

$\therefore$ $u_1 - y_2$ and $u_2 - y_1$ should work

c) We seek W_1 and W_2 such that

$$W_2 G W_1 = \begin{bmatrix} \tilde{G}_{11} & 0 \\ 0 & \tilde{G}_{22} \end{bmatrix}$$

Try $W_2 = I$ and

$$G(s)W_1(s) = \begin{bmatrix} \frac{5}{1+s} & \frac{-2}{1+s} \\ \frac{10}{2+s} & \frac{1}{2+s} \end{bmatrix}\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

Off-diagonals equal to zero gives

$$5w_{12} - 2w_{22} = 0$$
$$10w_{11} + w_{21} = 0$$

Thus, we can choose $w_{12} = 1 \Rightarrow w_{22} = 2.5$

$w_{11} = 1 \Rightarrow w_{21} = -10$

The transfer functions the PI controller should be designed for are

$$\tilde{G}_{11}(s) = \frac{5w_{11} - 2w_{21}}{1+s} = \frac{25}{1+s}$$

$$\tilde{G}_{22}(s) = \frac{10w_{12} + w_{22}}{2+s} = \frac{12.5}{2+s}$$