CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday August 25, 2023

Time and place:	08:30 - 12:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

Mobile telephones, laptops or tablets/iPads, are not allowed!

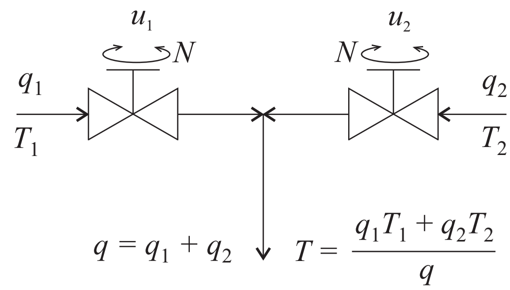
1. When using water from a tap you are interested in two properties: the flow q and the temperature T. A classical tap has two actuators (valves), one for cold water and one for hot water. Clearly, this is a coupled MIMO system because, for example, opening the hot water valve more will increase the flow as well as the temperature.

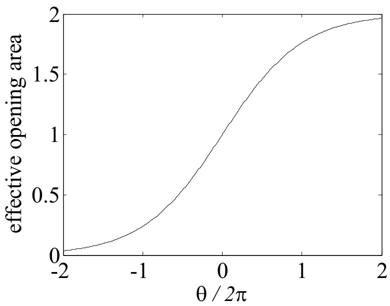
Classical two-grip tap

Modern one-grip tap

A modern one-grip tap is decoupled such that one input (vertical angle of the arm) affects the flow, and the other input (horizontal angle) affects the temperature.

Now, we have an industrial system with large flows and two *identical* valves, and we want to use multivariable control to achieve the same decoupling without redesigning the plant. The valves are motorized such that the actuation is driven by supply voltages u_1 and u_2 (see left figure).





The flow through each valve is proportional to the effective opening area (a_i),

$$q_i(t) = k_q a_i(t), \quad i = 1, 2 \tag{1}$$

where k_q is a constant. The effective opening area is a nonlinear function of the angle θ (rad) of the valve actuator (see right figure):

$$a_i(t) = 1 + \tanh\left(\frac{\theta_i(t)}{2\pi N}\right), \quad \text{where} \quad \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} \tag{2}$$

Here, N is the number of turns between closed and open valve. The relation between input voltage to each valve and the angle of the valve is

$$\frac{d}{dt}\theta_i(t) = k_\theta u_i(t), \quad i = 1, 2$$

where k_θ is a constant.

Note: This problem contains many questions. However, many of them can be solved independently of each other, so please read carefully through all questions!

(a) Assume constant temperatures T_1 and T_2 and determine a state space model for this system, with the flow q and the temperature T as outputs.

2 p.

(b) The cold temperature is $T_1 = 20°C$ and the hot temperature is $T_2 = 100°C$. Let $k_q = 1$, $k_\theta = 1$ and $2\pi N = 10$ and show that a linearized model for the operating point where $\theta_1 = \theta_2 = 0$ is

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} \Delta q \\ \Delta T \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} x(t)\end{aligned}$$

Help: $(d/dz) \tanh z = 1 - \tanh^2 z$

3 p.

(c) Show that for a sampling time $h = 0.1s$ and piecewise constant control signal, the corresponding discrete time model is

$$\begin{aligned}x(k+1) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} x(k)\end{aligned}$$

2 p.

(d) If we measure only the temperature T, can we estimate changes in the flow?

1 p.

(e) Suppose we measure both temperature and flow. Show that the discrete time transfer function for the system is

$$H(q) = \begin{bmatrix} \dfrac{0.01}{q-1} & \dfrac{0.01}{q-1} \\ \dfrac{-0.2}{q-1} & \dfrac{0.2}{q-1} \end{bmatrix},$$

where q is the time shift operator.

1 p.

(f) Decouple this MIMO system completely with a compensator W such that WH is diagonal.

1 p.

(g) Next, two SISO proportional controllers, $\tilde{H}_{c1} = K_1$ and $\tilde{H}_{c2} = K_2$ are chosen for the decoupled system WH (you need not carry out this design!). Express the MIMO controller from y_1 and y_2 to u_1 and u_2 in terms of the two SISO controllers and your elements in W. Give the elements in your controller matrix *or* show in a SISO-signal block diagram.

2 p.

2. Assume we have a system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + v_1(k) \\ y(k) &= Cx(k) + v_2(k) \end{aligned}$$

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sim WGN(0, \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix})$.

For this system the stationary Kalman filter (predictor case) is

$$\hat{x}(k+1|k) = A\hat{x}(k|k-1) + Bu(k) + K(y(k) - C\hat{x}(k|k-1))$$

The task is to show that for a Kalman filter it is mainly the ratio between the state variance R_1 and the measurement noise R_2 that determines the observer properties.

Assume that P^* is the estimation error variance for the case when $R_1 = R_1^*$ and $R_2 = R_2^*$. Show that the poles of the filter are unchanged if the ratio between the variances are unchanged, i.e. if $R_1 = \alpha R_1^*$ and $R_2 = \alpha R_2^*$.

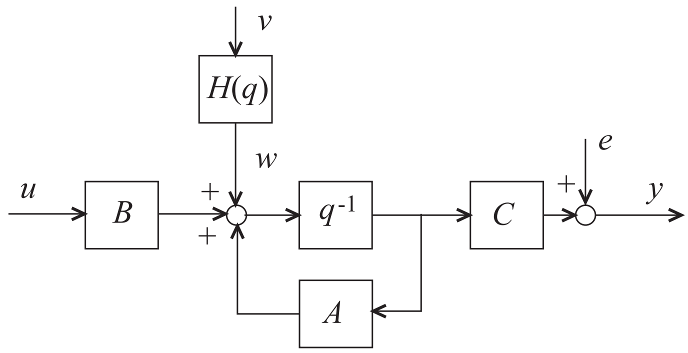
3 p.

3. Assume we have a system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + e(k) \end{aligned}$$

where e is a measurement white gaussian noise (WGN).

The process disturbance w is not white, though it originates from a source v that can be considered as WGN, see the figure below.



Neither w nor v can be measured online to be used in a feed forward control. However, from separate system identification experiments the pulse transfer operator H from v to w has been determined to be

$$H = \frac{\beta q^{-1}}{1 + \alpha_1 q^{-1} + \alpha_2 q^{-2}}$$

This can be used in an LQG controller by extending the state-space model and then estimate and feed back all states, i.e. the original process states and the disturbance states.

(a) Determine a state space model

$$\begin{aligned} x_w(k+1) &= A_w x_w(k) + B_w v(k) \\ w(k) &= C_w x_w(k) \end{aligned}$$

corresponding to H.

2 p.

(b) Give an extended state space model of the entire system, including the disturbance states, on the correct form for design of a Kalman filter (you need not calculate the filter).

2 p.

(c) Now, designing an LQ controller for the system we need to formulate the cost function. If we are only interested in controlling the output y and the two control signals are of equal kind, what cost matrices do you suggest?

1 p.

4. A linear time invariant system is given by

$$\begin{aligned}\frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0.5 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)\end{aligned}$$

(a) We only measure x_2 and x_3. Can x_1 be estimated?

2 p.

(b) Show that the system is not controllable?

1 p.

(c) Is the system stabilizable?

2 p.

(d) The eigenvectors of the system matrix (A) are

$$v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Diagonalize the system. *You may then use*

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

3 p.

(e) What happens to the uncontrollable state, and what will the consequences be on the original states x_1, x_2, and x_3?

1 p.

(f) Use the result from (e) to give a reduced state-space model, valid after initial transients have settled and keeping the meaning and quantities from the original model.

1 p.

Good luck!