Solutions to SSY285 230825

1 (a)

$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \underbrace{\begin{bmatrix}k_\theta & 0\\ 0 & k_\theta\end{bmatrix}}_{B}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\underline{q} = \underline{q}_1 + \underline{q}_2 = k_q\left(1+\tanh\frac{\theta_1}{2\pi N}\right) + k_q\left(1+\tanh\frac{\theta_2}{2\pi N}\right)$$

$$= k_q\left(2+\tanh\frac{\theta_1}{2\pi N} + \tanh\frac{\theta_2}{2\pi N}\right) = g_1(\theta_1,\theta_2)$$

$$T = \frac{\underline{q}_1 T_1 + \underline{q}_2 T_2}{\underline{q}}$$

$$= \frac{k_q}{\underline{q}(\theta_1,\theta_2)}\left(\left(1+\tanh\frac{\theta_1}{2\pi N}\right)T_1 + \left(1+\tan\frac{\theta_2}{2\pi N}\right)T_2\right)$$

$$= g_2(\theta_1,\theta_2)$$

(b)

$$\begin{cases} x_1 = \theta_1 - \bar\theta_1 = \theta_1\\ x_2 = \theta_2 - \bar\theta_2 = \theta_2\end{cases}$$

$$\Rightarrow \quad \dot x = Ax + Bu \quad \text{as above}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}\Delta \underline{q}\\ \Delta T\end{bmatrix} = \begin{bmatrix}\frac{\partial g_1}{\partial\theta_1} & \frac{\partial g_1}{\partial\theta_2}\\ \frac{\partial g_2}{\partial\theta_1} & \frac{\partial g_2}{\partial\theta_2}\end{bmatrix}_{(\bar\theta_1,\bar\theta_2)} x$$

$$\underline{q} = 2 + \tanh\frac{\theta_1}{10} + \tanh\frac{\theta_2}{10} = g_1$$

$$T = \frac{1}{\underline{q}}\left\{\left(1+\tanh\frac{\theta_1}{10}\right)T_1 + \left(1+\tanh\frac{\theta_2}{10}\right)T_2\right\} = g_2$$

$$\left.\frac{\partial g_1}{\partial\theta_1}\right|_{(\bar\theta_1,\bar\theta_2)} = \left.\frac{\partial \underline{q}}{\partial\theta_1}\right|_{(\bar\theta_1,\bar\theta_2)} = \frac{1}{10}\left(1-\tanh^2\frac{\bar\theta_1}{10}\right) = 0.1$$

$$\left.\frac{\partial g_1}{\partial \theta_2}\right|_{(\bar\theta_1, \bar\theta_2)} = \ldots = 0.1$$

$$\frac{\partial g_2}{\partial \theta_1} = \left\{ \frac{d}{dx} \frac{b(x)}{a(x)} = \frac{b'a - a'b}{v^2} \right\}$$

$$= \left\{ \begin{array}{l} b = (\quad) T_1 + (\quad) T_2 \\ a = g \end{array} \right\}$$

$$= \frac{1}{g^2} \left\{ \frac{T_1}{10} \left(1 - \tanh^2 \frac{\theta_1}{10}\right) g - \frac{1}{10} \left(1 - \tanh^2 \frac{\theta_1}{10}\right) \cdot b \right\}$$

$$\bar\theta_1 = \bar\theta_2 = 0 \Rightarrow \bar g = 2$$

$$\bar b = \bar T_1 + \bar T_2$$

$$\left.\frac{\partial g_2}{\partial \theta_1}\right|_{(\bar\theta_1, \bar\theta_2)} = \frac{1}{4} \left\{ 0.2 \bar T_1 - 0.1 (\bar T_1 + \bar T_2) \right\} = \frac{\bar T_1 - \bar T_2}{40}$$

Symmetry implies

$$\left.\frac{\partial g_2}{\partial \theta_2}\right|_{(\bar\theta_1, \bar\theta_2)} = \frac{\bar T_2 - \bar T_1}{40}$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c)

zero order hold discretization

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = C x(k)$$

$$\Phi = e^{Ah} = I + Ah + \ldots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Gamma = \int_0^h e^{A\tau} d\tau \, B = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$$

$$\begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} \Delta q(k) \\ \Delta T(k) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \end{bmatrix}$$

(d)

$C = [-2 \;\; 2]$ (only temp measured)

Observability matrix

$$W_0 = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \text{ has rank } 1 < 2$$

$\therefore$ Not observable $\Rightarrow$ we cannot estimate q

(e)

Discrete time transfer function

$$H(z) = C[zI - \Phi]^{-1}\Gamma$$

$$= \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

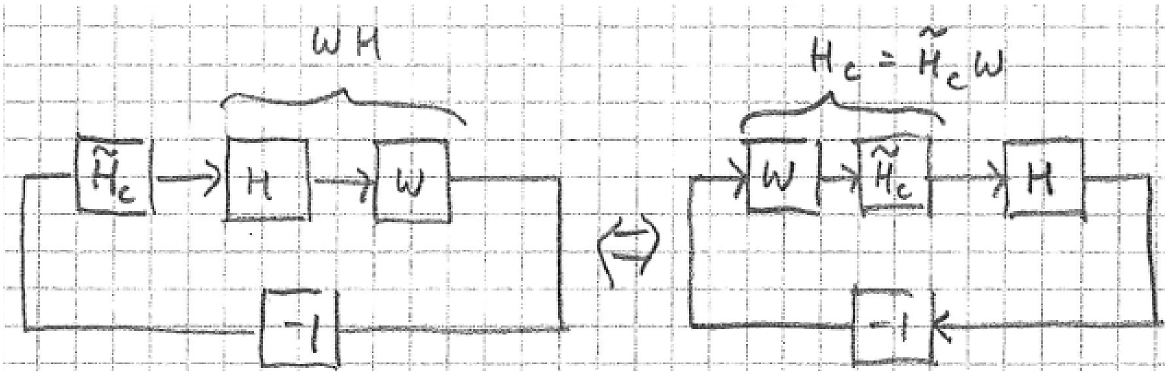
$$= \frac{1}{z-1} \begin{bmatrix} 0.01 & 0.01 \\ -0.2 & 0.2 \end{bmatrix}$$

(f)

WH diagonal if, e.g. $W = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} 5 & -0.25 \\ 5 & 0.25 \end{bmatrix}$$

(g)



Controller $H_c = \begin{bmatrix} \tilde{H}_{c1} & 0 \\ 0 & \tilde{H}_{c2} \end{bmatrix} \begin{bmatrix} 5 & -0.25 \\ 5 & 0.25 \end{bmatrix}$

$$= \begin{bmatrix} 5\tilde{H}_{c1} & -0.25\tilde{H}_{c1} \\ 5\tilde{H}_{c2} & 0.25\tilde{H}_{c2} \end{bmatrix}$$

2.

The poles of the stationary Kalman filter are given by the eigenvalues of

$A - KC$

where

(1) $K = APC^T(R_2 + CPC^T)^{-1}$

(2) $P = APA^T + R_1 - APC^T(R_2 + CPC^T)^{-1}CPA^T$

K unchanged $\Rightarrow$ poles unchanged

Let K^* and P^* be the solution for $R_1 = R_1^*$ and $R_2 = R_2^*$

$\Rightarrow P^* = AP^*A^T + R_1^* - AP^*C^T(R_2^* + CP^*C^T)^{-1}CP^*A^T$

Multiply (2) by α

$\alpha P^* = A\alpha P^* A^T + \alpha R_1^* - A\alpha P^* \underbrace{(\qquad)^{-1} CP^*A^T}_{(\alpha R_2^* + C\alpha P^* C^T)^{-1} C\alpha P^* A^T}$

$\therefore$ $P = \alpha P^*$ is the solution when $R_1 = \alpha R_1^*$ and $R_2 = \alpha R_2^*$

$\Rightarrow K = A\alpha P^* C^T(\alpha R_2^* + C\alpha P^* C^T)^{-1}$

$= AP^*C^T(R_2^* + CP^*C^T)^{-1} = K^*$

Thus, the poles are unchanged

3 (a)

$$w(k) = \frac{\beta q^{-1}}{1 + \alpha_1 q^{-1} + \alpha_2 q^{-2}} v(k)$$

$$w(k) + \alpha_1 w(k-1) + \alpha_2 w(k-2) = \beta v(k-1)$$

$$w(k+1) + \alpha_1 w(k) + \alpha_2 w(k-1) = \beta v(k)$$

$$\underbrace{\begin{bmatrix} w(k+1) \\ w(k) \end{bmatrix}}_{x_w(k+1)} = \underbrace{\begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix}}_{A_w} \underbrace{\begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}}_{x_w(k)} + \underbrace{\begin{bmatrix} \beta \\ 0 \end{bmatrix}}_{B_w} v(k)$$

$$w(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_w} \begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}$$

(b)

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ w(k+1) \\ w(k) \end{bmatrix}}_{x_e(k+1)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -\alpha_1 & -\alpha_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ w(k) \\ w(k-1) \end{bmatrix}}_{x_e(k)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(k)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \beta \\ 0 \end{bmatrix} v(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_e(k) + e(k)$$

(c)

A cost function without cross terms is suggested, i.e.

$$V = \sum x_e^T Q_x x_e + u^T Q_u u$$

Since the control signals are of equal kind, a suitable initial cost matrix would be

$$Q_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the extended state vector we should not have any costs on the disturbance states. Now, this does not matter here since we only want to control $y = Cx_e$

$$y^T y = x_e^T C^T C^T x_e$$

and thus we should use a cost matrix

$$Q_x = C^T C \cdot q = q \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Where q is a tuning parameter

4 (a)

$$A = \begin{bmatrix} -1 & 0.5 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \left\{ A^2 = \begin{bmatrix} 1.25 & -1 & 0 \\ -1 & 1.25 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & -1 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix} \Big\} \; n = 3 \text{ linearly indep rows}$$

$\therefore$ System observable, i.e. all states can be estimated.

(b)

Controllability matrix

$$S = [B \; AB \; A^2B] = \begin{bmatrix} 0 & 1 & 0 & -0.5 & 0 & 2.25 \\ 0 & 1 & 0 & -0.5 & 0 & 2.25 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

Two first rows are the same and the third cannot be 'scaled' from the others

$\Rightarrow$ rank $(S) = 2$

$\therefore$ not controllable

(c)

A stable system is always stabilizable.
Let us check the eigenvalues!

$$\det(\lambda I - A) = (\lambda+1)((\lambda+1)^2 - 0.25) = 0$$

$$(\lambda+1) = 0 \Rightarrow \lambda = -1 \in LHP$$

$$(\lambda+1)^2 - 0.25 = 0 \Rightarrow \lambda = -1 \pm 0.5 = \begin{cases} -0.5 \\ -1.5 \end{cases} \in LHP$$

System stable, thus stabilizable.

(d)

Definition of eigenvalue and eigenvector

$$Av_i = \lambda_i v_i \Rightarrow \begin{bmatrix} -1 & 0.5 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1.5 & & 0 \\ & -1 & \\ 0 & & -0.5 \end{bmatrix}$$

$$AV = V\Lambda, \ \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \Rightarrow V^{-1}AV = \Lambda$$

$$z = Tx \Rightarrow \dot{z} = T\dot{x} = T(Ax + Bu)$$

$$= \underbrace{TAT^{-1}}_{A_z} z + \underbrace{TB}_{B_z} u$$

with $T = V^{-1}$ we get

$$\dot{z} = \underbrace{\begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}}_{A_z} z + \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{B_z} u$$

$$B_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$y = Cx = CT^{-1}z = CVz$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} z$$

(e)

From the diagonalized system we see that z_1 is the uncontrollable state, i.e. we cannot affect z_1 with u since the corresponding elements in B_z are zero and the states do not affect eachother in a diagonalized system.

Since the transformed and original system have the same eigenvalues z_1 is stable

$\Rightarrow z_1 \to 0$ as $t \to \infty$

$z = V^{-1}x \Rightarrow z_1 = \frac{1}{\sqrt{2}}(x_1 - x_2)$

$\Rightarrow x_1 \to x_2$ as $t \to \infty$

(f)

Replacing x_1 by x_2 the original model becomes

$$\begin{cases} \dot{x}_2 = -0.5x_2 + u \\ \dot{x}_3 = -x_3 + u \end{cases} \Rightarrow \dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{cases} y_1 = x_2 \\ y_2 = x_3 \end{cases} \qquad y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

(Noise ignored)