CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday April 3, 2023

Time and place:	08:30 - 12:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

Mobile telephones, laptops or tablets/iPads, are not allowed!

1. Consider a linear quadratic optimization problem where the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is to be controlled such that

$$V = \int_0^\infty x^T(t)Q_x x(t) + u^T(t)Q_u u(t)dt$$

is minimized.

(a) For an optimal control law $u(t) = -Lx(t)$ to exist, Q_x and Q_u must be positive semidefinite. Why is that requirement needed?

1 p.

(b) In addition, Q_u is actually required to be positive finite. Please give an explanation to that as well.

1 p.

2. Consider the following multivariable system

$$\begin{aligned} \frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

First, the system was to be optimally controlled such that

$$V = \int_0^\infty x^T(t)Q_x x(t) + u^T(t)Q_u u(t)dt$$

is minimized. However, it turns out that the stationary accuracy of the controlled outputs y is not sufficent, and therefore integral action is introduced.

Show how this can be achieved using LQR and state the equations (you need not solve them) that must be solved. Also, explain how you can increase or decrease the integral action.

3 p.

3. A state x is the output of interest, of a first order process with white noise v as input. x is then measured by m identical sensors not affecting each other, each having a measurement noise that can be considered as white, i.e.,

$$\begin{aligned} \frac{d}{dt}x(t) &= ax(t) + v(t), \qquad E\{v(t)v(\tau)\} = q\delta(t-\tau) \\ y_i(t) &= x(t) + w_i(t), \qquad E\{w_i(t)w_i(\tau)\} = r\delta(t-\tau), \quad i = 1, 2 \dots m \end{aligned}$$

where $\delta(t)$ is a Dirac pulse.

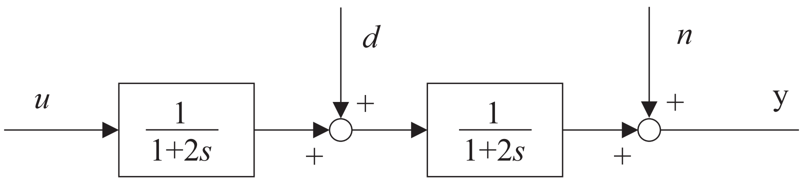
(a) Since we know a we can apply a Kalman filter to estimate x, i.e. the optimal estimate of x. What is the ratio between the estimation error variance when using m sensors and the estimation error variance when using only one sensor?

3 p.

(b) Let $q = r = 1$ and $m = 10$, but calculate the ratio between the estimation error variance when $a = -1$ and when $a = -0.1$. What is your explanation why the variance is higher for $a = -0.1$ than for $a = -1$?

2 p.

4. Consider the block scheme below.



We can assume the disturbance n is white noise with unit variance, and for d the following spectrum has been determined experimentally:

$$\Phi_d(\omega) = \frac{1}{\omega^2 + 1}$$

(a) Write the system on the state space form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Ne(t) \\ y(t) &= Cx(t) + n(t), \end{aligned}$$

where $e(t)$ is white noise with intensity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2 p.

(b) Assume we have determined the spectrum also for the measurement noise n as

$$\Phi_n(\omega) = \frac{\omega^2 + 4}{\omega^2 + 9}.$$

Rewrite the state space model once more on standard form (Note! Not the same matrices as in (a))

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nv_1(t)\\ y(t) &= Cx(t) + v_2(t),\end{aligned}$$

where $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is white noise with intensity $R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$!

2 p.

(c) What will the intensity matrix R be if n, d and v are all independent?

2 p.

5. For a discrete time controllable system with n states a *deadbeat controller* takes the system from any state to the origin in at most n samples. This is achieved by designing the state feedback such that all closed loop poles becomes zero. Analogously, a *deadbeat observer* is an observer that will have zero observer error after at most n samples, given that the model is correct and there are no disturbances.

Consider

$$\begin{aligned}x(k+1) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)\\ y(k) &= \begin{bmatrix} 2 & 0 \end{bmatrix} x(k)\end{aligned}$$

and the observer

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K(y(k) - C\hat{x}(k))$$

(a) Can the observer poles be placed arbitrarily for this system?

1 p.

(b) For a deadbeat observer all poles are zero. Determine a deadbeat observer for the above system.

2 p.

(c) First show that the estimation error indeed becomes zero in 2 samples. Then use Cayley-Hamilton's theorem to show that if all observer poles (eigenvalues) are zero, then the estimation error for an LTI-system with n states must be zero after n samples.

3 p.

6. In a continuous bioreactor bacteria (for example yeast) are fed a substrate (for example sugar). The equations governing this process are (derived by mass balances w.r.t. bacteria and substrate, respectively)

$$
\begin{aligned}
V\frac{d}{dt}S(t) &= -\frac{\mu}{Y}X(t)\frac{S(t)}{S(t)+k} - QS(t) + QS_{feed}(t) \\
V\frac{d}{dt}X(t) &= \mu X(t)\frac{S(t)}{S(t)+k} - bX(t) - QX(t) + QX_{feed}(t)
\end{aligned}
$$

where $V = 1$ is the reactor volume, $X(t)$ the bacterial concentration, $S(t)$ the substrate concentration, $Q = 0.1$ is the flow through the reactor, $\mu = 1$ the maximum growth rate of the bacteria, $k = 1$ a saturation coefficient, $b = 0.9$ the death rate of the bacteria, $Y = 0.2$ the yield coefficient, $X_{feed}(t)$ the bacteria concentration in the influent and $S_{feed}(t)$ the substrate concentration in the influent, which we can regard as our input control signal $u(t)$.

(a) Let x_1 be the substrate concentration, x_2 the bacterial concentration, v be the bacterial concentration in the feed (X_{feed}, a disturbance), and write the equations on the standard state-space model form $\dot{x} = f(x, u, v)$

1 p.

(b) Linearize the model, assuming we have an equilibrium at the operating point given by a desired effluent bacterial concentration $\bar{X} = 5$ and an influent bacterial concentration $\bar{X}_{feed} = 10$. Show that the resulting model becomes:

$$
\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1.1 & -4 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \Delta u + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \Delta v
$$

3 p.

(c) Suppose we can measure both concentrations in the tank, i.e. x_1 and x_2. Design a state feedback controller with both eigenvalues in $\lambda = -1$.

3 p.

(d) Can we expect a faster or a slower control if we move the eigenvalues to $\lambda = -10$ (give a short motivation)?

1 p.

Good luck!