1a) If $Q_u$ is not positive semidefinite, then there are vectors $u$ for which $u^T Q_u u < 0$ and

$\min_u \int u^T Q_u u \rightarrow -\infty$

Similarly, if $\exists\, x : x^T Q_x x < 0$ then a minimum of $V$ will not give $x \rightarrow 0$

b) The Riccati equation that has to be solved has a term with $Q_u^{-1}$, which consequently requires $Q_u$ to be definite (otherwise not invertible)

(Intuition also says that there has to be a cost on control)

2) I $\dot{x} = Ax + Bu$

$y = Cx$

Ⓘ Introduce integral action by adding an integral state

$$x_I = \int_0^t (r(\tau) - y(\tau))\,d\tau \Rightarrow \dot{x}_I = r - y = r - Cx$$

Thus the extended system is

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ x_I \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_e} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

where

$$A_e = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Ⅱ Modify the criterion to be minimized

$$V = \int_0^t x_e^T Q_{xe} x_e + u^T Q_u u \, dt$$

where $Q_{xe} = \begin{bmatrix} Q_x & 0 \\ 0 & Q_I \end{bmatrix}$

Ⅲ Solve the Riccatiequation w.r.t. $S$ and $L$

$$A_e^T S + S A_e + Q_{xe} - S B_e Q_u^{-1} B_e^T S = 0$$

$$L = Q_u^{-1} B_e^T S$$

Ⅳ Integral action is increased by increasing $Q_I$ relative to $Q_x$ and $Q_u$

3 a) The estimation error variance for the Kalman filter, $P = Var\{\hat{x} - x\}$ is given by the solution to the c.t. Riccati equation (5.79) with $R_1 = \underline{q}$, $R_2 = r \cdot I_m$, $R_{12} = 0$, $N = 1$, and $C = [1\ 1 \ldots 1]^T$ (i.e. $m$ equal sensors)

i.e.

$$2aP - P[1 \ldots 1] r \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} P + \underline{q} = 0$$

$$2aP - \frac{P^2 m}{r} + \underline{q} = 0$$

$$P^2 - \frac{2ar}{m} P = \frac{\underline{q} r}{m}$$

$$\left(P - \frac{ar}{m}\right)^2 = \frac{\underline{q} r}{m} \overset{+}{(-)} \left(\frac{ar}{m}\right)^2 \qquad \{P > 0\}$$

$$= \frac{r(a^2 r + \underline{q} m)}{m^2}$$

$$\Rightarrow P(m) = \frac{ar}{m} + \frac{1}{m}\sqrt{r(a^2 r + \underline{q} m)}$$

$$\frac{P(m)}{P(1)} = \frac{ar + \sqrt{r(a^2 r + \underline{q} m)}}{m\left(ar + \sqrt{r(a^2 r + \underline{q})}\right)}$$

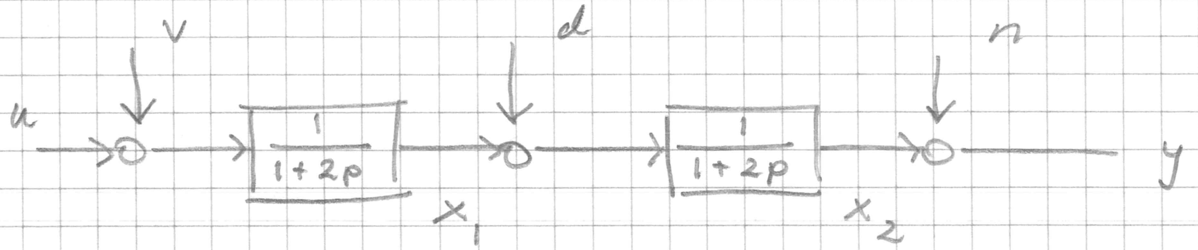
b) $\underline{q} = r = 1$ and $m = 10$ give

$$P(a) = \frac{1}{10}\left(a + \sqrt{a^2 + 10}\right)$$

$P(-0.1) = 0.3064$, $P(-1) = 0.23$

The larger $|a|$ is the more important the dynamics/model is $\Rightarrow$ Kalman improves

4)




a)

With $x_1$ and $x_2$ as above

$$x_1 = \frac{1}{1+2p}(u+v) \Leftrightarrow \dot{x}_1 = -\frac{1}{2}x_1 + \frac{1}{2}u + \frac{1}{2}v$$

$$x_2 = \frac{1}{1+2p}(x_1+d) \Leftrightarrow \dot{x}_2 = -\frac{1}{2}x_2 + \frac{1}{2}x_1 + \frac{1}{2}d$$

Spectral factorization

$$\Phi_d(\omega) = \frac{1}{\omega^2+1} \Leftrightarrow d(t) = \frac{1}{p+1}\nu(t)$$

if $\nu(t)$ white noise with unit intensity

Let $d = x_3 \Rightarrow \dot{x}_3 = -x_3 + \nu$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \nu \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + n(t)$$

where $e = \begin{bmatrix} v \\ \nu \end{bmatrix}$ have intensity $\begin{bmatrix} \Phi_v & 0 \\ 0 & \Phi_\nu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $$\Phi_n = \frac{\omega^2 + 4}{\omega^2 + 9}$$

Spectral factorization again gives

$$n = \frac{p+2}{p+3} v_2 \quad , \quad v_2 \text{ white noise with unit intensity}$$

$$n = \frac{p+3}{p+3} v_2 - \underbrace{\frac{1}{p+3} v_2}_{x_4} = v_2 - x_4$$

$$\Rightarrow \quad \dot{x}_4 = -3 x_4 + v_2$$

$$\therefore \quad \dot{x} = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} v \\ \nu \\ v_2 \end{bmatrix}}_{v_1}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} x + v_2$$

c) $v, d, n$ independent $\Rightarrow$
$v, \nu, v_2$ — " —

$$\Rightarrow \quad R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} \Phi_{v v_2} \\ \Phi_{\nu v_2} \\ \Phi_{v_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad R_2 = \Phi_{v_2} = 1$$

$$\therefore \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

5 a) The observer poles can be placed arbitrarily for an LTI system if it is observable.

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

has rank 2 $\Rightarrow$ observable $\Rightarrow$ $\underline{\underline{\text{Yes}}}$

b)
$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$
$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K(Cx(k) - C\hat{x}(k))$$

The estimation error is given by

$$\tilde{x}(k+1) = x(k+1) - \hat{x}(k+1)$$
$$= (A - KC)\tilde{x}(k)$$
$$= \left( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix} \right) \tilde{x}(k)$$
$$= \begin{bmatrix} 1-2k_1 & 1 \\ 1-2k_2 & 0 \end{bmatrix} \tilde{x}(k)$$

The observer poles equals the eigenvalues of $A - KC$, i.e.

$$\det(\lambda I - A + KC) = \det \begin{bmatrix} \lambda + 2k_1 - 1 & -1 \\ 2k_2 - 1 & \lambda \end{bmatrix}$$
$$= \lambda^2 + (2k_1 - 1)\lambda + 2k_2 - 1 = (\lambda - p_1)(\lambda - p_2) = 0$$

If both poles are zero, i.e. $p_1 = p_2 = 0$ then

$$\left.\begin{aligned} 2k_1 - 1 &= 0 \\ 2k_2 - 1 &= 0 \end{aligned}\right\} \Rightarrow K = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

c) With this $K$ we get

$$(A - KC) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Assume $\tilde{x}(k) \neq 0$

$$\tilde{x}(k+1) = (A - KC)\tilde{x}(k)$$

$$\tilde{x}(k+2) = (A - KC)\tilde{x}(k+1) = (A - KC)^2 \tilde{x}(k)$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \tilde{x}(k) = 0$$

$\therefore$ The estimation error is zero 2 samples later

Now, let $A_0 = A - KC$ for a general observable LTI system, and $K$ be chosen such that all eigenvalues are zero.

CH $\Leftrightarrow$ A matrix satisfies its own characteristic equation, i.e.

$$\det(\lambda I - A_0) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$$

If all poles are in $\lambda = 0$ then the char. eqn must be $p(\lambda) = \lambda^n = 0$. CH then gives

$$p(A_0) = A_0^n = 0$$

$$\Rightarrow \tilde{x}(k+n) = A_0 \tilde{x}(k+n-1) = \dots = A_0^n \tilde{x}(k) = 0$$

6 a)

$$\begin{cases} \dot{x}_1 = -5x_2 \dfrac{x_1}{x_1+1} - 0.1x_1 + 0.1u \equiv f_1 \\[2ex] \dot{x}_2 = x_2 \dfrac{x_1}{x_1+1} - x_2 + 0.1v \equiv f_2 \end{cases}$$

b) The above equations are only nonlinear in $x_1$ and $x_2$ and not in $u$ and $v$.

$$\frac{\partial f_1}{\partial x_1} = -5x_2 \cdot \frac{x_1+1-x_1}{(x_1+1)^2} - 0.1 = -\frac{5x_2}{(x_1+1)^2} - 0.1$$

$$\frac{\partial f_1}{\partial x_2} = -\frac{5x_1}{x_1+1}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{x_2}{(x_1+1)^2}$$

$$\frac{\partial f_2}{\partial x_2} = \frac{x_1}{x_1+1} - 1$$

In the operating point $f_1 = f_2 = 0$, $\bar{x}_2 = 5$, $\bar{v} = 10$

$f_2(\bar{x}, \bar{u}, \bar{v}) = 0$ gives

$$\frac{5\bar{x}_1}{\bar{x}_1+1} - 4 = 0 \Rightarrow \bar{x}_1 = 4$$

Then $f_1(\bar{x}, \bar{u}, \bar{v}) = 0$ gives

$$-25\frac{4}{5} - 0.4 + 0.1\bar{u} = 0 \Rightarrow \bar{u} = 204$$

With $\Delta x_1 = x_1 - \bar{x}_1$, $\Delta x_2 = x_2 - \bar{x}_2$, $\Delta u = u - \bar{u}$, $\Delta v = v - \bar{v}$ we have

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}_{(\bar{x},\bar{u},\bar{v})} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix} \Delta u + \begin{bmatrix} \partial f_1/\partial v \\ \partial f_2/\partial v \end{bmatrix} \Delta v$$

$$= \begin{bmatrix} -1.1 & -4 \\ 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \Delta u + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \Delta v$$

c) With state feedback $u = -[l_1 \ \ l_2]\Delta x$ the closed loop poles are the eigenvalues of $A - BL$

$$\det(\lambda_I - A + BL) = \det\left\{ \begin{bmatrix} \lambda+1.1 & 4 \\ -0.2 & \lambda+0.2 \end{bmatrix} + \begin{bmatrix} 0.1l_1 & 0.1l_2 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \det \begin{bmatrix} \lambda+1.1+0.1l_1 & 4+0.1l_2 \\ -0.2 & \lambda+0.2 \end{bmatrix}$$

$$= \lambda^2 + \lambda(1.3+0.1l_1) + 0.2(1.1+0.1l_1) + 0.2(4+0.1l_2)$$

$$= \lambda^2 + \lambda(1.3+0.1l_1) + 1.02 + 0.02(l_1+l_2)$$

$$\equiv P(\lambda)$$

To get double poles in $\lambda = -1$ we must have

$$P(\lambda) = (\lambda+1)^2 = \lambda^2 + 2\lambda + 1$$

Thus

$$\left.\begin{aligned} 1.3+0.1l_1 &= 2 \\ 1.02+0.02(l_1+l_2) &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} l_1 &= 7 \\ l_2 &= \frac{-0.02}{0.02} - 7 = -8 \end{aligned}$$

d) The eigenvalues determines how quickly transients pass ($e^{\lambda t}$). The further left in LHP the faster the system

$\therefore \ \lambda_{1,2} = -10$ instead of $-1$ $\Rightarrow$ faster