1 a) Newtons law: $ma = \sum F$

$$\Rightarrow m\ddot{z} = F - F_w = u - 400v^2$$

Let $x_1 = z$ and $x_2 = v = \dot{z}$. This gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(u - 400x_2^2) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Let $\Delta x(t) = x(t) - \bar{x}$, $\Delta u(t) = u(t) - \bar{u}$, and $\Delta y(t) = y(t) - \bar{y}$, where $\bar{x}, \bar{u}, \bar{y}$ is the equilibrium operating point where the speed $\bar{v} = \bar{x}_2 = 5$ m/s. The linearized model then becomes

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}_{(\bar{x},\bar{u})} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{df_1}{du} \\ \frac{df_2}{du} \end{bmatrix}_{(\bar{x},\bar{u})} \Delta u$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & \frac{800\bar{x}_2}{m} \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \Delta u$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0,4 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ 10^{-4} \end{bmatrix}}_{B} \Delta u$$

$$\Delta y = \Delta x = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \Delta x$$

c) Discretization 'zoh' with $h = 0.01\,s$:

$$\Delta x(k+1) = A_d\, \Delta x(k) + B_d\, \Delta u(k)$$
$$\Delta y(k) = C\, \Delta x(k)$$

$$A_d = e^{Ah} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}_{t=h}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+0.4 \end{bmatrix}^{-1} = \frac{1}{s(s+0.4)} \begin{bmatrix} s+0.4 & 1 \\ 0 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+0.4)} \\ 0 & \frac{1}{s+0.4} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}_{t=h} = \begin{bmatrix} 1 & 2.5(1 - e^{-0.4h}) \\ 0 & e^{-0.4h} \end{bmatrix}$$

$$h = 0.01 \Rightarrow \underline{\underline{A_d = \begin{bmatrix} 1 & 0.01 \\ 0 & 0.996 \end{bmatrix}}}$$

$$B_d = \int_0^h e^{At} B\, dt = \int_0^h \begin{bmatrix} 2.5 \cdot 10^{-4} (1 - e^{-0.4t}) \\ 10^{-4} \cdot e^{-0.4t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 2.5 \cdot 10^{-4} (h + 2.5(e^{-0.4h} - 1)) \\ -2.5 \cdot 10^{-4} (e^{-0.4h} - 1) \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 5.0 \cdot 10^{-9} \\ 1.0 \cdot 10^{-6} \end{bmatrix}}}$$

2 $A = \begin{bmatrix} -1 & 0.5 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) Observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \left\{ A^2 = \begin{bmatrix} 1.25 & -1 & 0 \\ -1 & 1.25 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & -1 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix} \Big\} \; n = 3 \text{ linearly indep rows}$$

$\therefore$ System observable, i.e. all states can be estimated.

b) Controllability matrix

$$S = [B \; AB \; A^2B] = \begin{bmatrix} 0 & 1 & 0 & -0.5 & 0 & 2.25 \\ 0 & 1 & 0 & -0.5 & 0 & 2.25 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

Two first rows are the same and the third cannot be 'scaled' from the others

$\Rightarrow$ rank$(S) = 2$

$\therefore$ not controllable

c) A stable system is always stabilizable.
Let us check the eigenvalues!

$$\det(\lambda I - A) = (\lambda+1)\left((\lambda+1)^2 - 0.25\right) = 0$$

$$(\lambda+1) = 0 \Rightarrow \lambda = -1 \in LHP$$

$$(\lambda+1)^2 - 0.25 = 0 \Rightarrow \lambda = -1 \pm 0.5 = \begin{cases} -0.5 \\ -1.5 \end{cases} \in LHP$$

System stable, thus stabilizable.

d) Definition of eigenvalue and eigenvector

$$Av_i = \lambda_i v_i \Rightarrow \begin{bmatrix} -1 & 0.5 & 0 \\ 0.5 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1.5 & & 0 \\ & -1 & \\ 0 & & -0.5 \end{bmatrix}$$

$$AV = V\Lambda, \ \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \Rightarrow V^{-1}AV = \Lambda$$

$$z = Tx \Rightarrow \dot{z} = T\dot{x} = T(Ax + Bu)$$
$$= \underbrace{TAT^{-1}}_{A_z} z + \underbrace{TB}_{B_z} u$$

with $T = V^{-1}$ we get

$$\dot{z} = \underbrace{\begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}}_{A_z} z + \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{B_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}} u$$

$$y = Cx = CT^{-1}z = CVz$$
$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} z$$

e) From the diagonalized system we see that $z_1$ is the uncontrollable state, i.e. we cannot affect $z_1$ with $u$ since the corresponding elements in $B_z$ are zero and the states do not affect eachother in a diagonalized system.

Since the transformed and original system have the same eigenvalues $z_1$ is stable

$$\Rightarrow z_1 \to 0 \text{ as } t \to \infty$$

$$z = V^{-1}x \quad \Rightarrow \quad z_1 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

$$\Rightarrow \underline{x_1 \to x_2 \text{ as } t \to \infty}$$

f) Replacing $x_1$ by $x_2$ the original model becomes

$$\begin{cases} \dot{x}_2 = -0.5x_2 + u \\ \dot{x}_3 = -x_3 + u \end{cases} \Rightarrow \quad \dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{cases} y_1 = x_2 \\ y_2 = x_3 \end{cases} \qquad y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

(Noise ignored)

3) The Kalman decomposition implies that only states that are both observable and controllable are needed to describe the dynamics from $u$ to $y$

$n = 12$ and

$\text{rank}(S) = 8 \Rightarrow$ 8 states are controllable
$\text{rank}(\mathcal{O}) = 7 \Rightarrow$ 7 states are observable

The smallest union of the two is e.g.

$$x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8 \; x_9 \; x_{10} \; x_{11} \; x_{12}$$

controllable  observable

The largest union is if all observable states are controllable.

a) 3 states

b) 7 states

4 a) All step responses looks like step responses of 1st order systems with no time delay, i.e.

$$G_{ij}(s) = \frac{K_{ij}}{1+sT_{ij}} \quad , \; i,j = 1,2$$

The gains $K_{ij}$ are the final value (since step size is 1) and the time constants are the times when appr. 63 % of the change has occurred.

This gives

$$G_{11}(s) \approx \frac{5}{1+s} \qquad G_{12} \approx \frac{2}{1+s}$$

$$G_{21}(s) \approx \frac{7.5}{1+0.5s} = \frac{15}{2+s}$$

$$G_{22}(s) \approx \frac{0.5}{1+0.5s} = \frac{1}{2+s}$$

$$\Rightarrow G(s) = \begin{bmatrix} \frac{5}{s+1} & \frac{2}{s+1} \\ \frac{15}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 5(s+2) & 2(s+2) \\ 15(s+1) & (s+1) \end{bmatrix}$$

b) $RGA(G(j\omega)) = G(j\omega) .* G^{-T}(j\omega)$

$$G^{-1}(s) = \frac{1}{25} \begin{bmatrix} -(s+1) & 2(s+2) \\ 15(s+1) & -5(s+2) \end{bmatrix}$$

$$G(s) .* G^{-T}(s) = \begin{bmatrix} -\frac{5}{25} & \frac{30}{25} \\ \frac{30}{25} & -\frac{5}{25} \end{bmatrix} = \begin{bmatrix} -0.2 & 1.2 \\ 1.2 & -0.2 \end{bmatrix}$$

Thus RGA does not depend on $\omega$ in this case.

Negative diagonal elements $\Rightarrow$ we should avoid diagonal pairing

Off diagonal elements close to 1

$\Rightarrow$ Choose pairing $y_1 \leftrightarrow u_2$
$y_2 \leftrightarrow u_1$

Alt. from plot only

From plot we see that

$$G(0) = \begin{bmatrix} 5 & 2 \\ 7.5 & 0.5 \end{bmatrix}$$

$$RGA(G(0)) = G(0) .* G^{-T}(0)$$

$$= \begin{bmatrix} 5 & 2 \\ 7.5 & 0.5 \end{bmatrix} .* \frac{1}{2.5-15} \begin{bmatrix} 0.5 & -7.5 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 & 1.2 \\ 1.2 & -0.2 \end{bmatrix}$$

Avoid diagonal coupling $\Rightarrow$ only option is

$y_1 \leftrightarrow u_2$
$y_2 \leftrightarrow u_1$

5 a) $\dot{x} = -x + u + w$

$$\Phi_w = \frac{1}{1+\omega^2}$$

Spectral factorization gives that we can construct disturbance $w$ as

$$v_1 \longrightarrow \boxed{G} \longrightarrow w$$

where $v_1 \sim WN(0,1)$ and $G(j\omega)G(-j\omega) = \Phi_w(\omega)$

$$\Phi_w(\omega) = \frac{1}{(1+j\omega)}\frac{1}{(1-j\omega)}$$

$$\Rightarrow \quad w(t) = \frac{1}{1+p} v_1(t)$$

$$(1+p)w = w + \dot{w} = v_1$$

Extended state space model:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ w \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_e} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{N_e} v_1$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_e} \begin{bmatrix} x \\ w \end{bmatrix} + v$$

where $v_2 \sim WN(0,1)$ and $v_1$ & $v_2$ are independent ($\Rightarrow R_1 = 1,\ R_2 = 1,\ R_{12} = 0$)

The optimal observer (minimizing the variance of the estimation error) is the continuous time Kalman filter, i.e.

$$\dot{\hat{x}}_e = A_e x_e + B_e u + K(y - C_e \hat{x}_e)$$

where $K$ is the solution to (Theorem 5.4)

$$K = PC_e^T R_2^{-1} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{1} = \underline{\underline{\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}}}$$

$$AP + PA^T - PC_e^T R_2^{-1} (PC_e^T)^T + N_e R_1 N_e^T = 0$$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} [p_{11} \ p_{12}]$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ 1] = 0$$

$$\left.\begin{aligned} (1,1)&: \quad (-p_{11} + p_{12})2 - p_{11}^2 = 0 \\ (1,2)&: \quad -p_{12} + p_{22} - p_{12} - p_{11}p_{12} = 0 \\ (2,2)&: \quad -p_{22} \cdot 2 - p_{12}^2 + 1 = 0 \end{aligned}\right\} \Rightarrow P$$

choose solution where $P > 0$

This eqn system is too hard to solve by hand. However, it was stated that

$$p_{11} \equiv \text{Var}\{\hat{x} - x\} = 0.2$$

$$(1,1) \Rightarrow \quad p_{12} = p_{11} + \frac{1}{2} p_{11}^2 = 0,2 + \frac{0.04}{2} = 0.22$$

$$\Rightarrow \quad K = \begin{bmatrix} 0.2 \\ 0.22 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.2 \\ 0.22 \end{bmatrix} \left( y - [1 \ 0] \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1.2 & 1 \\ -0.22 & -1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.2 \\ 0.22 \end{bmatrix} y$$

5 b) $w$ scalar and can be modelled using spectral factorization means that $w$ can be modelled as the output of a stable transfer fcn $G(s) = \frac{B(s)}{A(s)}$

Written on observer canonical form this transfer fcn can be realized as

$$\begin{bmatrix} \dot{w} \\ \dot{x}_w \end{bmatrix} = \underbrace{\begin{bmatrix} -a_1 & 1 & 0 & \cdots \\ -a_2 & 0 & 1 & \\ \vdots & & & \ddots \end{bmatrix}}_{A_w} \begin{bmatrix} w \\ x_w \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}}_{B_w} v_1 \ , \ v_1 \sim WN$$

Adding this to the original model

$$\begin{bmatrix} \dot{x} \\ \dot{w} \\ \dot{x}_w \end{bmatrix} = \underbrace{\begin{bmatrix} A & N & 0 & \cdots & 0 \\ 0 & [ & A_w & ] \end{bmatrix}}_{A_e} \begin{bmatrix} x \\ w \\ x_w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \underbrace{\begin{bmatrix} 0 \\ B_w \end{bmatrix}}_{N_e} v_1$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_e} \begin{bmatrix} x \\ w \\ x_w \end{bmatrix} + v_2$$

Requirements for $\exists$ Kalman filter are stated in Lemma 5.1. For this problem they are

(i) $R_1 \geq 0$ (Obviously fullfilled)

(ii) $(A_e, C_e)$ detectable
(iii) $(A_e, R_1)$ stabilizable
} Fullfilled if $A_e$ is stable

$A$ stable, $A_w$ stable $\det(\lambda I - A_e) =$
$= \det(\lambda I - A)\det(\lambda I - A_w) \Rightarrow A_e$ stable

$\therefore$ Yes, one can always determine a KF!