1) $x(k+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$

$u(k) = -Lx(k) + Kr(k) = -[l_1 \; l_2]x(k) + Kr(k)$

$\Rightarrow x(k+1) = \underbrace{\begin{bmatrix} 1-l_1 & -l_2 \\ 1 & 1 \end{bmatrix}}_{A_c} x(k) + \underbrace{\begin{bmatrix} K \\ 0 \end{bmatrix}}_{B_r} r(k)$

a) Poles $\equiv$ eigenvalues of A_c

$$\det(\lambda I - A_c) = \det \begin{bmatrix} \lambda - 1 + l_1 & l_2 \\ -1 & \lambda - 1 \end{bmatrix}$$

$$= (\lambda-1)(\lambda-1-l_1) + l_2$$

$$= \lambda^2 + \lambda(l_1 - 2) + l_2 - l_1 + 1$$

$$= (\lambda - 0.9)^2$$

$$= \lambda^2 - 1.8\lambda + 0.81$$

$\Rightarrow$ $\underline{\underline{l_1 = 0.2}}$ and $l_2 = 0.81 + l_1 - 1 = \underline{\underline{0.01}}$

b) $x(k+1) = A_c x(k) + B_r r(k)$

$y(k) = Cx(k)$

Stable & SS $\Rightarrow x(k+1) = x(k) = \bar{x}$

$(I - A_c)\bar{x} = B_r \bar{r}$

$\Rightarrow \bar{y} = C(I - A_c)^{-1} B_r \bar{r}$

$$= [0 \; 1] \begin{bmatrix} l_1 & l_2 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} K \\ 0 \end{bmatrix} \bar{r}$$

$$= \underbrace{[0 \; 1] \frac{1}{l_2} \begin{bmatrix} 0 & -l_2 \\ 1 & l_1 \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix}}_{=1} \bar{r}$$

$\Rightarrow 100K = 1 \Rightarrow \underline{\underline{K = 0.01}}$

2) If reachable then poles can be placed arbitrarily

Controllability matrix

$$\mathcal{C} = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

full rank $\Rightarrow$ reachable

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3) $$G(p) = \begin{bmatrix} 0 & \frac{2}{p+1} \\ 0 & \frac{1}{p} \\ \frac{1}{p+1} & \frac{2}{p+1} \end{bmatrix}$$

a) Largest minors (order 2)

$$0 \;, \; -\frac{1}{p(p+1)} \;, \; \frac{-2}{(p+1)^2}$$

Order 1 = elements of G

The polepolynomial is the least common divisor of all minors, i.e.

$$P(p) = p(p+1)^2$$

$\therefore$ 3 poles : $0, -1, -1$

The minors are consequently

$$\frac{0}{P(p)}, \quad \frac{p+1}{P(p)}, \quad \frac{2p}{P(p)}$$

Zero polynomial is the largest common divisor, which is 1 in this case

$\Rightarrow$ No zeros

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b) $y_1 = \frac{2}{p+1} u_2$

$y_2 = \frac{1}{p} u_2$

$y_3 = \frac{1}{p+1} u_1 + \frac{2}{p+1} u_2$

Let, for example

$x_1 = \frac{1}{p+1} u_2, \quad x_2 = \frac{1}{p} u_2, \quad x_3 = \frac{1}{p+1} u_1$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Minimal since 3 states and 3 poles

(observable and controllable equivalently)

4) a) $y(k) = \dfrac{1}{q - 0.5} u(k) \Rightarrow y(k+1) - 0.5 y(k) = u(k)$

$\therefore$
$$x(k+1) = 0.5 x(k) + u(k)$$
$$y(k) = x(k)$$

b) $$x_I(k) = \frac{1}{q-1}(r(k) - y(k))$$

$$\Rightarrow \quad x_I(k+1) - x_I(k) = r(k) - x(k)$$

$\therefore$
$$\underbrace{\begin{bmatrix} x(k+1) \\ x_I(k+1) \end{bmatrix}}_{x_e(k+1)} = \underbrace{\begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(k) \\ x_I(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_e} u(k) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{K_r} r(k)$$

$$y(k) = \underbrace{[1 \quad 0]}_{C_e} x_e(k)$$

c) Optimal control law (Assume $r = 0$ and $L_e = [l_1\ l_2]$)

$$u(k) = -L_e x_e(k)$$
$$= -l_1 x(k) - l_2 x_I(k)$$
$$= -l_1 y(k) + l_2 \frac{1}{q-1} y(k)$$

Block scheme gives

$$u(k) = -\left(K_p + K_I \frac{1}{q-1}\right) y(k)$$

$\therefore$ $K_p = l_1$ and $K_I = -l_2$

d) The obtained LQ controller guarantees a stable system

5) $$G(s) = \frac{1}{s} e^{-0.6s}$$

Time delay $T_d = 0.6$, Sampling time $h = 1$

We recognise this process as a pure integration ($1/s$) with a time delay, i.e.

a) $$\dot{x}(t) = a x(t) + b u^*(t) \quad , \quad u^*(t) = u(t - 0.6)$$

$$y(t) = c x(t)$$

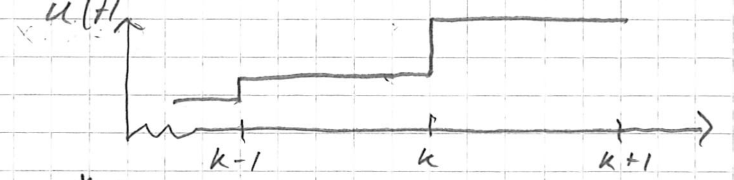
with $a = 0$, $b = 1$, $c = 1$

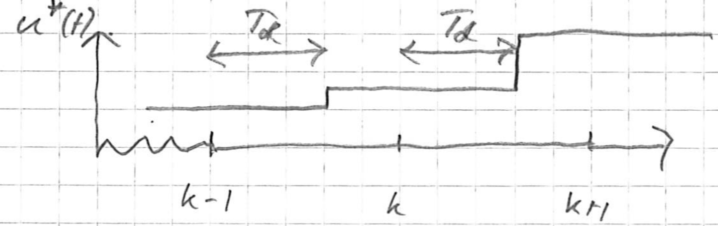
b) Given the state x_0 at time t_0 the analytical solution is

$$x(t) = e^{a(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} B u^*(\tau) \, d\tau$$

Let $t_0 = kh = k$ and $t = (k+1)h = k+1$.

Study input signal on the interval





Consequently, splitting the interval into two

$$x(k+1) = x(k) + \int_{k}^{k+T_d} B u(k-1) \, d\tau + \int_{k+T_d}^{k+1} B u(k) \, d\tau$$

$$x(k+1) = x(k) + T_d u(k-1) + (1-T_d)u(k)$$

Introduce delayed input state

$$x_d(k) = u(k-1)$$

Then with $T_d = 0.6$

$$\begin{bmatrix} x(k+1) \\ x_d(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x_d(k) \end{bmatrix} + \begin{bmatrix} 0.4 \\ 1 \end{bmatrix} u(k)$$

6)
$$x(k+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(k) + v_1(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(k) + v_2(k)$$

$$v_1 \sim WGN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) \Rightarrow R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$v_2 \sim WGN(0, 1) \Rightarrow R_2 = 1$$

$$E\{v_1 v_2\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow R_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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a)
$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\text{rank}(\mathcal{O}) = 2 \Rightarrow$ observable

b) Standard Kalman (predictor) in SS:

$$\hat{x}_p(k+1) = A\hat{x}_p(k) + Bu(k) + K(y(k) - C\hat{x}_p(k))$$

where $\hat{x}_p(k) = \hat{x}(k|k-1)$

$\text{Var}\{x_p(k)\} = P$ where P is given by

$$P = APA^T + R_1 - K(CPA^T + R_{12}^T)$$

$$K = (APC^T + R_{12})(R_2 + CPC^T)^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(1 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} p_{12} \\ 0 \end{bmatrix} \frac{1}{1 + p_{11}}$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} p_{22} & 0 \\ 0 & 0 \end{bmatrix}} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$- \frac{1}{1+p_{11}} \underbrace{\begin{bmatrix} p_{12} \\ 0 \end{bmatrix} [1 \quad 0] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} p_{12}^2 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$\Rightarrow \left.\begin{aligned} p_{11} &= p_{22} + 1 - \frac{p_{12}^2}{1+p_{11}} \\ p_{12} &= 0 \\ p_{22} &= 2 \end{aligned}\right\} P = \underline{\underline{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}} \Rightarrow K = \underline{\underline{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}}$$

c) Filter case

(1) $\hat{x}_f(k+1) = \hat{x}_p(k+1) + K_f(y(k) - C\hat{x}_p(k))$

where $\hat{x}_f(k) = \hat{x}(k|k)$ and

(2) $\hat{x}_p(k+1) = A\hat{x}_f(k) + Bu(k)$

$$K_f = PC^T(CPC^T + R_2)^{-1}$$

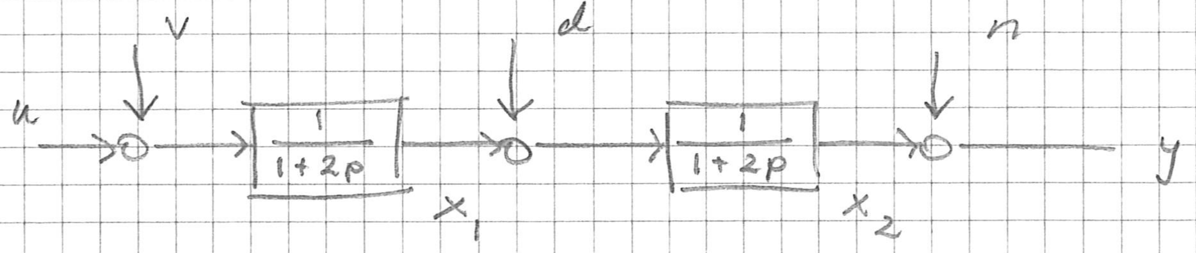
$$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{1+p_{11}} = \underline{\underline{\begin{bmatrix} 3/4 \\ 0 \end{bmatrix}}}$$

$$\mathrm{Var}\{\hat{x}_f\} = P_f = P - PC^T(CPC^T + R_2)^{-1}CP$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 3/4 \\ 0 \end{bmatrix} [1 \quad 0] \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0 & 2 \end{bmatrix}$$

$\mathrm{Var}\{\hat{x}_2\}$ unchanged but $\mathrm{Var}\{\hat{x}_1\}$ reduced by 75 %

7)



a)

With x_1 and x_2 as above

$$x_1 = \frac{1}{1+2p}(u+v) \iff \dot{x}_1 = -\frac{1}{2}x_1 + \frac{1}{2}u + \frac{1}{2}v$$

$$x_2 = \frac{1}{1+2p}(x_1+d) \iff \dot{x}_2 = -\frac{1}{2}x_2 + \frac{1}{2}x_1 + \frac{1}{2}d$$

Spectral factorization

$$\Phi_d(\omega) = \frac{1}{\omega^2+1} \iff d(t) = \frac{1}{p+1}\nu(t)$$

if $\nu(t)$ white noise with unit intensity

Let $d = x_3 \Rightarrow \dot{x}_3 = -x_3 + \nu$

$\therefore$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \nu \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + n(t)$$

where $e = \begin{bmatrix} v \\ \nu \end{bmatrix}$ have intensity $\begin{bmatrix} \Phi_v & 0 \\ 0 & \Phi_\nu \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $$\Phi_n = \frac{\omega^2+4}{\omega^2+9}$$

Spectral factorization again gives

$$n = \frac{p+2}{p+3} v_2 \quad , \quad v_2 \text{ white noise with unit intensity}$$

$$n = \frac{p+3}{p+3} v_2 - \underbrace{\frac{1}{p+3} v_2}_{x_4} = v_2 - x_4$$

$$\Rightarrow \quad \dot{x}_4 = -3x_4 + v_2$$

$$\therefore \quad \dot{x} = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} v \\ \nu \\ v_2 \end{bmatrix}}_{v_1}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} x + v_2$$

c) v, d, n independent $\Rightarrow$
v, ν, v_2 — " —

$$\Rightarrow \quad R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} \Phi_{v v_2} \\ \Phi_{\nu v_2} \\ \Phi_{v_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad R_2 = \Phi_{v_2} = 1$$

$$\therefore \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$