LCSD 190424

1

a)

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k_\theta & 0 \\ 0 & k_\theta \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underline{q} = \underline{q}_1 + \underline{q}_2 = k_q\left(1 + \tanh\frac{\theta_1}{2\pi N}\right) + k_q\left(1 + \tanh\frac{\theta_2}{2\pi N}\right)$$

$$= k_q\left(2 + \tanh\frac{\theta_1}{2\pi N} + \tanh\frac{\theta_2}{2\pi N}\right) = g_1(\theta_1, \theta_2)$$

$$T = \frac{\underline{q}_1 T_1 + \underline{q}_2 T_2}{\underline{q}}$$

$$= \frac{k_q}{\underline{q}(\theta_1, \theta_2)}\left(\left(1 + \tanh\frac{\theta_1}{2\pi N}\right)T_1 + \left(1 + \tan\frac{\theta_2}{2\pi N}\right)T_2\right)$$

$$= g_2(\theta_1, \theta_2)$$

b)

$$\begin{cases} x_1 = \theta_1 - \bar{\theta}_1 = \theta_1 \\ x_2 = \theta_2 - \bar{\theta}_2 = \theta_2 \end{cases}$$

$\Rightarrow$ $\dot{x} = Ax + Bu$ as above

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta \underline{q} \\ \Delta T \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} \end{bmatrix}\right|_{(\bar{\theta}_1, \bar{\theta}_2)} x$$

$$\underline{q} = 2 + \tanh\frac{\theta_1}{10} + \tanh\frac{\theta_2}{10} = g_1$$

$$T = \frac{1}{\underline{q}}\left\{\left(1 + \tanh\frac{\theta_1}{10}\right)T_1 + \left(1 + \tanh\frac{\theta_2}{10}\right)T_2\right\} = g_2$$

$$\left.\frac{\partial g_1}{\partial \theta_1}\right|_{(\bar{\theta}_1, \bar{\theta}_2)} = \left.\frac{\partial \underline{q}}{\partial \theta_1}\right|_{(\bar{\theta}_1, \bar{\theta}_2)} = \frac{1}{10}\left(1 - \tanh^2\frac{\bar{\theta}_1}{10}\right) = 0.1$$

cont'd

$$\left.\frac{\partial g_x}{\partial \theta_2}\right|_{(\bar{\theta}_1,\bar{\theta}_2)} = \ldots = 0.1$$

$$\frac{\partial g_2}{\partial \theta_1} = \left\{ \frac{d}{dx}\frac{b(x)}{a(x)} = \frac{b'a - a'b}{v^2} \right\}$$

$$= \left\{ \begin{array}{l} b = (\quad)T_1 + (\quad)T_2 \\ a = g \end{array} \right\}$$

$$= \frac{1}{g^2}\left\{ \frac{T_1}{10}\left(1-\tanh^2\frac{\theta_1}{10}\right)g - \frac{1}{10}\left(1-\tanh^2\frac{\theta_1}{10}\right)\cdot b \right\}$$

$$\bar{\theta}_1 = \bar{\theta}_2 = 0 \Rightarrow \bar{g} = 2$$

$$\bar{b} = \bar{T}_1 + \bar{T}_2$$

$$\left.\frac{\partial g_2}{\partial \theta_1}\right|_{(\bar{\theta}_1,\bar{\theta}_2)} = \frac{1}{4}\left\{0.2\bar{T}_1 - 0.1(\bar{T}_1 + \bar{T}_2)\right\} = \frac{\bar{T}_1 - \bar{T}_2}{40}$$

Symmetry implies

$$\left.\frac{\partial g_2}{\partial \theta_2}\right|_{(\bar{\theta}_1,\bar{\theta}_2)} = \frac{\bar{T}_2 - \bar{T}_1}{40}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c) zero order hold discretization

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = C x(k)$$

$$\Phi = e^{Ah} = I + Ah + \ldots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Gamma = \int_0^h e^{A\tau} d\tau B = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$$

$$\begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \end{bmatrix} + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} \Delta q(k) \\ \Delta T(k) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \end{bmatrix}$$

d) Controllability matrix

$$W_c = [\Gamma \;\; \Phi\Gamma] = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 \end{bmatrix}$$

which has full rank = 2 since column 2 cannot be a constant times column 1.

$\Rightarrow$ reachable

e) $C = [-2 \;\; 2]$ (only temp measured)

Observability matrix

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \text{ has rank } 1 < 2$$

$\therefore$ Not observable $\Rightarrow$ we cannot estimate q

f) Discrete time transfer function

$$H(z) = C[zI - \Phi]^{-1}\Gamma$$

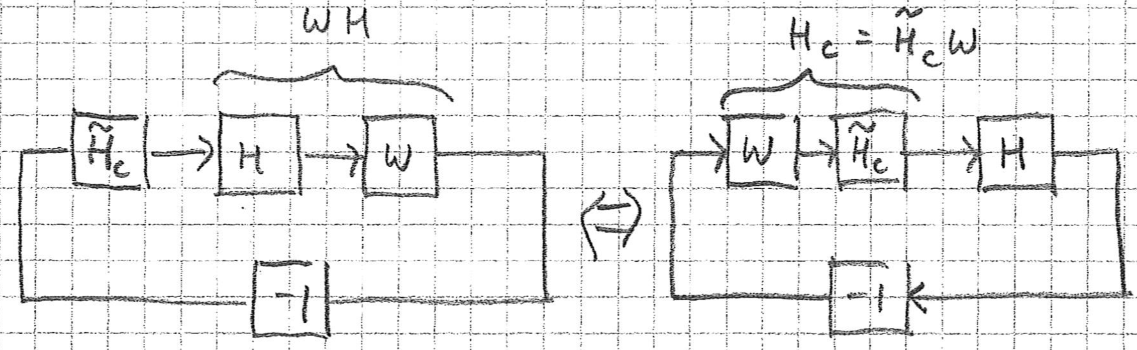
$$= \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$= \frac{1}{z-1} \begin{bmatrix} 0.01 & 0.01 \\ -0.2 & 0.2 \end{bmatrix}$$

g) WH diagonal if, e.g. $W = \begin{bmatrix} 0.1 & 0.1 \\ -2 & 2 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} 5 & -0.25 \\ 5 & 0.25 \end{bmatrix}$$

h)



Controller $$H_c = \begin{bmatrix} \tilde{H}_{c1} & 0 \\ 0 & \tilde{H}_{c2} \end{bmatrix} \begin{bmatrix} 5 & -0.25 \\ 5 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 5\tilde{H}_{c1} & -0.25\tilde{H}_{c1} \\ 5\tilde{H}_{c2} & 0.25\tilde{H}_{c2} \end{bmatrix}$$

2) $x(k+1) = 0.8x(k) + v_1(k), \quad v_1 \sim WGN(0, R_1)$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(k) + v_2(k), \quad v_2 \sim WGN(0, R_2)$$

$$\Rightarrow A = 0.8, \quad B = 0, \quad N = 0, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = 0$$

$$R_1 = 1, \quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R_{12} = [0 \ 0] \quad \text{(indep)}$$

a) Minimum variance estimate is given by the stationary Kalman filter (predictor case)

$$\hat{x}(k+1) = 0.8\hat{x}(k) + K\left(y(k) - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\hat{x}(k)\right)$$

$$K = APC^T(CPC^T + R_2)^{-1} = 0.8P[1 \ 1]\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}P[1 \ 1] + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^{-1}$$

$$= 0.8P[1 \ 1]\begin{bmatrix} P+1 & P \\ P & P+2 \end{bmatrix}^{-1}$$

$$= \frac{0.8P}{3P+2}[1 \ 1]\begin{bmatrix} P+2 & -P \\ -P & P+1 \end{bmatrix} = \frac{0.8P}{3P+2}[2 \ 1]$$

where the estimation error variance is

$$P = APA^T + R_1 - \underbrace{APC^T(CPC^T + R_2)^{-1}}_{K} CPA^T$$

$$= 0.64P + 1 - \frac{0.8P}{3P+2}[2 \ 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} 0.8P$$

$$0 = (1 - 0.36P)(3P+2) - 3 \cdot 0.64P^2$$

$$= 2 + 2.28P - 3P^2$$

$$\left(P - \frac{1.14}{3}\right)^2 = \frac{2}{3} + \left(\frac{1.14}{3}\right)^2 = 0.811$$

$$\Rightarrow P = \underline{\underline{1.28}} \quad (-0.52 \text{ incorrect since } P > 0)$$

$$\Rightarrow K = \underline{\underline{0.175[2 \ 1]}}$$

b) Solution given by the filter case

$$\hat{x}(k/k) = \hat{x}(k) + \tilde{K}(y(k) - C\hat{x}(k))$$

$$\tilde{K} = PC^T(CPC^T + R_2)^{-1}$$

$$= p[1\ 1]\frac{1}{3p+2}\begin{bmatrix} p+2 & -p \\ -p & p+1 \end{bmatrix}$$

$$= \frac{p}{3p+2}[2\ 1] = 0.22\,[2\ 1]$$

The variance of this estimate is

$$P(k/k) = P - \underbrace{PC^T(CPC^T + R_2)^{-1}}_{\tilde{K}}CP =$$

$$= 1{,}28(1 - \underbrace{3 \cdot 0.22}_{0.66 = \underline{\underline{66\ \%}}}) = 0.44$$

3a) $Y_1 = \frac{1}{4+s} u \Rightarrow \dot{y}_1 + 4y_1 = u$

$Y = \frac{1}{1+s} y_1 \Rightarrow \dot{y} + y = y_1$

Let $x_1 = y$ and $x_2 = y_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $x_I = \int_0^t e(\tau)\, d\tau \Rightarrow \dot{x}_I = r - y = r - Cx$
$= r - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Extended state space model

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_I \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{A_e} \begin{bmatrix} x_1 \\ x_2 \\ x_I \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{B_e} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_e} x_e$$

c) $J = \int_0^\infty y^2 + u^2 + q_I x_I^2 \, dt$

$$= \int_0^\infty \begin{bmatrix} x_1 & x_2 & x_I \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q_I \end{bmatrix}}_{Q_x} \begin{bmatrix} x_1 \\ x_2 \\ x_I \end{bmatrix} + u \cdot \underbrace{1}_{Q_u} \cdot u \, dt$$

The controller that minimizes J is the LQR solution

$$u = -L\, x_e$$

where L_e is given by the solution to

$$L = Q_u^{-1} B_e^T S \qquad (Q_{xu} = [0 \;\; 0])$$

$$0 = A_e^T S + S A_e + Q_x - S B_e Q_u^{-1} (S B_e)^T$$

where $S_{3\times3}$ symmetric and $\succ 0$

d) From the block scheme we have

$$U(s) = K\left(-Y_1(s) - \left(K_p + \frac{K_I}{s}\right) Y(s)\right)$$

$$= -K Y_1(s) - K K_p Y(s) - \frac{K K_I}{s} Y(s)$$

which corresponds to

$$u(t) = -K x_2(t) - K K_p x_1(t) - K K_I x_I(t)$$

$$= -[l_1 \;\; l_2 \;\; l_3]\, x_e$$

$$\Rightarrow K = l_2, \quad K K_p = l_1 \Rightarrow K_p = \frac{l_1}{l_2}, \quad l_3 = K K_I \Rightarrow K_I = \frac{l_3}{l_2}$$

e) Increase the integral action, i.e.

Increase q_I and recalculate L

4 a) $$x(k+1) = \underbrace{\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{B} u(k)$$

Controllability matrix

$$W_c = [B \;\; AB \;\; A^2B] = \left\{ A^2 = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 4 & -2 \\ -2 & 0 & 10 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 8 \\ -2 & -6 & -20 \end{bmatrix}$$

W_c does not have full rank (3) since, e.g.

$4 \cdot$ column 1 $+ 2 \cdot$ column 2 $=$ column 3

However, it has more than rank 1 since

$\nexists$ a: a $\cdot$ column 1 = column 2

b) Repeated application of the state space equation gives ($n = \dim(A) = 3$)

$$x(n) = \underbrace{A^n x(0)}_{x(0)=0} + \underbrace{[B \;\; AB \; .. \; A^{n-1}B]}_{W_c} \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

We see that all reachable states $\in \mathcal{R}(W_c)$

Fundamental theorem of linear algebra

$\Rightarrow$ all unreachable states $\in \mathcal{N}(W_c^T)$

$$\begin{bmatrix} 0 & 1 & -2 \\ 2 & 2 & -6 \\ 4 & 8 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \Rightarrow x_2 = 2x_3 \\ \Rightarrow 2x_1 - 2x_3 = 0 \\ \Rightarrow 4x_1 - 4x_3 = 0 \end{matrix}$$

$\therefore$ States where $\begin{cases} x_2 = 2x_1 \\ x_3 = x_1 \end{cases}$

cannot be reached from the origin