CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday January 14, 2019

Time and place:	14:00 - 18:00 at Hörsalsvägen
Teacher:	Torsten Wik (031 - 772 5146)

Important!

There are two problem 4 (A and B) such that you can choose the one you prefer, depending on your background.

You should only solve one of them!

Further information

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. *Feedback Systems* by Åström and Murray OR *Reglerteknikens grunder* by Lennartson OR Reglerteknik, grundlaggande teori by Glad and Ljung OR *Linear optimal control systems* by H. Kwakernaak and R. Sivan

   Paper copies are accepted instead of books

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

Notes, mobile telephones, laptops or palmtops, are not allowed!

The results are open for review on Monday January 28, at 13:00 - 14:00 at the department.

Good Luck!

1. Assume that we have a process described by

$$
\begin{aligned}
x(k+1) &= ax(k) + bu(k) + v_1(k) \\
y(k) &= cx(k) + v_2(k)
\end{aligned}
$$

where v_1 and v_2 are independent white zero mean Gaussian noise with variances σ_1^2 and σ_2^2, respectively.

The system parameters are $a = 1/\sqrt{2}$, $b = 1$ and $c = 1/\sqrt{2}$.

(a) Determine the stationary observer giving the smallest achievable variance of the estimation error $x(k) - \hat{x}(k)$, where the estimate is based on the current measurement $y(k)$, i.e. $\hat{x}(k) = \hat{x}(k|k)$.

4 p.

(b) If disturbance and noise variances are $\sigma_1 = 1$ and $\sigma_2 = 1$, how large is the variance of the estimation error?

1 p.

(c) What happens to the pole of the filter when the measurement noise increases to become very large compared to σ_1^2? Give a "physical" interpretation.

3 p.

2. Assume we can model a continuous time system as

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + e(t)
\end{aligned}
$$

where w and e are stationary gaussian stochastic processes that are independent. The measurement noise e is white with intensity $R_e = 2$, but the process disturbance is not. Spectral measurements of this disturbance has given the spectrum

$$
\Phi_w(\omega) = \frac{1}{(4+\omega^2)(1+\omega^2)}
$$

Give all the matrices (and scalars) needed to calculate the optimal estimate of x.

3 p.

3. Consider the linear time invariant system

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t)
\end{aligned}
$$

In MATLAB we run the command (`A` is the system matrix and `C` is the measurement matrix above)

```
>> rank(obsv(A,C))

ans =

     2
```

from which we conclude (after checking more carefully) that the system is not observable.

We then run the command

```
>> [V,LAMBDA]=eig(A)

V =

         0    1.0000         0
    0.7071         0   -0.7071
    0.7071         0    0.7071


LAMBDA =

    -2     0     0
     0    -1     0
     0     0     0

>> inv(V)

ans =

         0    0.7071    0.7071
    1.0000         0         0
         0   -0.7071    0.7071
```

The columns of `V` are the eigenvectors of `A` where the diagonal elements of `LAMBDA` are the corresponding eigenvalues such that `A*V = V*LAMBDA`.

(a) Diagonalize the system, i.e. $z = Tx$ such that the new system matrix is diagonal, and show what linear combination z_i of x_1, x_2 and x_3 that is not observable.

3 p.

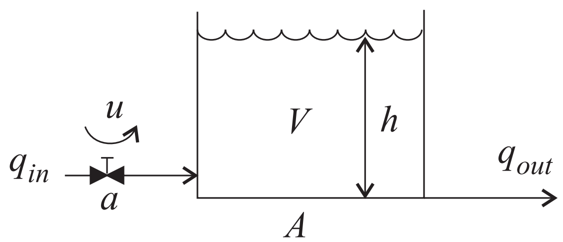
(b) What is the least number of states needed to describe how the inputs affect the outputs?

2 p.

Choose this question (4A) or the next (4B) depending on your preference/background. The applications are different but the problems are equivalent.

Note: Both problems contain many questions. However, many of them can be solved independently of each other, so please read carefully through all questions!

4A To compensate for variations in a flow (q_{out}) a buffer tank with bottom area A is used. The level h is measured and controlled by the inlet valve (see figure).



The flow through the valve is proportional to the opening area a of the valve according to

$$q_{in} = k_a a \sqrt{h_0 - h}$$

where k_a and h_0 are constants and $h_0 > \max h$.

The opening area is controlled by a motor which has the control input signal u. The opening area is proportional to the number of turns of the valve, which gives

$$a = k_u \int_0^t u(\tau) d\tau$$

where k_u is a constant.

(a) Determine a state space model for the system describing how the level h depends on the control signal u.

2 p.

(b) Let $k_a = k_u = V = A = 1$, $h_0 = 2$ and show that

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -0.05 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

where $x_1(t) = h(t) - \bar{h}$ and $x_2(t) = a(t) - \bar{a}$, approximates the system around the operating point $\bar{h} = 1$ m and $q_{out} = 0.1$ m^3/s.

2 p.

(c) The system is to be controlled at a sampling rate of 10Hz using a zero order hold on the control signal. Show that the corresponding discrete time state space model for the system is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned} \qquad (1)$$

3 p.

(d) Is the discrete time system (1) reachable?

1 p.

(e) Use (1) to determine the discrete time transfer function (operator) $G(q)$ from u to y.

2 p.

(f) It can be very important to know if a valve is saturated, i.e. entirely closed or completely open, since the operating range of the valve determines what flows that can actually be compensated for.

Suppose we only measure the level, can the valve opening area a be estimated then?

1 p.

(g) Suppose process and measurement noise are Gaussian and we estimate both the level ($\hat{x}_1$) and the valve opening area ($\hat{x}_2$) using a Kalman filter. Now, we want to minimize the sum of the variances of the sampled variables, i.e.

$$J = Var[h] + Var[a] + Var[u]$$

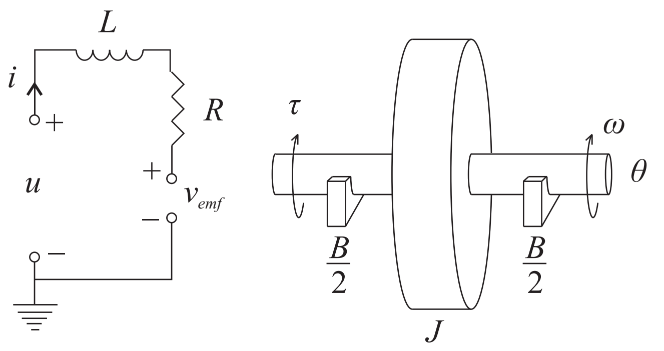
State the equations and give the matrices needed to determine the optimal state feedback $u(k) = -L\hat{x}(k)$, where $\hat{x}(k)$ is the estimate based on measurements up to time $k-1$. Note: You need not solve them!

2 p.

(h) A simulation of the system with the above controller shows that the variations in the level become to large, and that the actuation can be made far more aggressive. Suggest a modification of the criterion J that should improve the performance of the controlled system.

1 p.

4B Consider a DC motor with an inductance L and a resistance R applied to a load with inertia J (including the inertia of the motor).



The back voltage v_{emf} is assumed to be proportional to the angular velocity ω (rad/s),

$$v_{emf} = K_m \omega \quad (V),$$

the torque can be assumed to be proportional to the current, i.e.

$$\tau = K_\tau i \quad (Nm),$$

and the total friction can be assumed to be

$$\tau_f = B\sqrt{\omega} \quad (Nm)$$

(a) Determine a nonlinear state space model of the system describing how the current and motor speed ω depend on the applied voltage u.

2 p.

(b) Let $K_m = K_\tau = B = J = L = 1$, $R = 2$ and show that if only the current is measured

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -2 & -1 \\ 1 & -0.5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

where $x_1(t) = i(t) - \bar{i}$ and $x_2(t) = \omega(t) - \bar{\omega}$, approximates the system around the operating point $\bar{\omega} = 1$ rad/s.

2 p.

(c) This system is to be controlled from a control system working at 10Hz and with a piecewise constant control signal. Show that the corresponding discrete time state space model for the system is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.81 & -0.088 \\ 0.088 & 0.95 \end{bmatrix} x(k) + \begin{bmatrix} 0.090 \\ 0.0046 \end{bmatrix} u(k) \qquad (2) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned}$$

3 p.

(d) Is the discrete time system (2) reachable?

1 p.

(e) Use (2) to determine the discrete time transfer function (operator) $G(q)$ from u to y.

2 p.

(f) To save money we would like to avoid a motor speed indicator. If we measure only the current i, can we estimate the angular rate ω?

1 p.

(g) Suppose the process and measurement noise are Gaussian and we estimate both the current ($\hat{x}_1$) and the angular rate ($\hat{x}_2$) using a Kalman filter. Now, we want to minimize the sum of the variances of the sampled variables, i.e.

$$J = Var[i] + Var[\omega] + Var[u]$$

State the equations and give the matrices needed to determine the optimal state feedback $u(k) = -L\hat{x}(k)$, where $\hat{x}(k)$ is the estimate based on measurements up to time $k-1$. Note: You need not solve them!

2 p.

(h) A simulation of the system with the above controller shows that the variations in the motor speed become too large, and that the actuation can be made far more aggressive. Suggest a modification of the criterion J that should improve the performance of the controlled system.

1 p.