1 a) The optimal observer, which minimizes the variances is the discrete time stationary Kalman filter, filter case (Week 4)

$$\hat{x}(k|k) = \hat{x}(k) + \tilde{K}(y(k) - c\hat{x}(k))$$

where $\hat{x}(k)$ is the predictor case estimate

$$\hat{x}(k) = a\hat{x}(k-1) + bu(k-1) + K(y(k-1) - c\hat{x}(k-1))$$

$$\tilde{K} = PC^T(CPC^T + R_2)^{-1} = \frac{pc}{c^2p + \sigma_2^2}$$

$$K = (APC^T + N\underbrace{R_{12}}_{=0 \text{ because } v_1 \& v_2 \text{ independent}})(CPC^T + R_2)^{-1}$$

$$= \frac{apc}{c^2p + \sigma_2^2}$$

$$P = APA^T + NR_1N^T - K(APC^T + NR_{12})$$

$$= a^2p + \sigma_1^2 - \frac{a^2p^2c^2}{c^2p + \sigma_2^2} = 0.5p + \sigma_1^2 - \frac{0.25p}{0.5p + \sigma_2^2}$$

$$\Rightarrow 0.5p^2 + p\sigma_2^2 = \cancel{0.25p^2} + 0.5(\sigma_1^2 + \sigma_2^2)p + \sigma_1^2\sigma_2^2 - \cancel{0.25p^2}$$

$$0.5p^2 + 0.5p(\sigma_2^2 - \sigma_1^2) = \sigma_1^2\sigma_2^2$$

$$(p + 0.5(\sigma_2^2 - \sigma_1^2))^2 = 2\sigma_1^2\sigma_2^2 + 0.25(\sigma_1^4 + \sigma_2^4 + 2\sigma_1^2\sigma_2^2)$$

$$= 1.5\sigma_1^2\sigma_2^2 + 0.25(\sigma_1^4 + \sigma_2^4)$$

$$p = 0.5\left\{\sigma_1^2 - \sigma_2^2 \underset{p>0}{(\pm)} \sqrt{6\sigma_1^2\sigma_2^2 + \sigma_1^4 + \sigma_2^4}\right\}$$

$$\Rightarrow \tilde{K} = \frac{0.354\left(\sigma_1^2 - \sigma_2^2 + \sqrt{6\sigma_1^2\sigma_2^2 + \sigma_1^4 + \sigma_2^4}\right)}{0.25\left(\sigma_1^2 + \sqrt{6\sigma_1^2\sigma_2^2 + \sigma_1^4 + \sigma_2^4}\right) + 0.75\sigma_2^2}$$

$$\Rightarrow K = \frac{\tilde{K}}{\sqrt{2}}$$

b)

$$\left.\begin{array}{l}\sigma_1 = 1\\ \sigma_2 = 1\end{array}\right\} \Rightarrow \underline{\underline{P = \sqrt{2}}}$$

$$\tilde{K} = \frac{\sqrt{2}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{2}\sqrt{2} + 1} = \frac{2}{2+\sqrt{2}} = \underline{\underline{2-\sqrt{2}}}$$

$$K = \frac{\tilde{K}}{\sqrt{2}} = \frac{2}{2\sqrt{2}+2} = \frac{1}{1+\sqrt{2}} = \underline{\underline{\sqrt{2}-1}}$$

The variance of $x - \hat{x}(k|k)$ is

$$\tilde{P} = P - PC^T(CPC^T + R_2)^{-1}CP$$

$$= P\left(1 - \frac{c^2P}{c^2P + \sigma_2^2}\right) = \sqrt{2}\left(1 - \frac{0.5\sqrt{2}}{0.5\sqrt{2}+1}\right) = \underline{\underline{0.828}}$$

(a reduction of $(P-\tilde{P})/P = (\sqrt{2}-0.83)/\sqrt{2} = 42\%$!)

c) The pole of the Kalman filter is given by

$$\lambda - (a - Kc) = 0 \Rightarrow \lambda = \frac{1}{\sqrt{2}}\left(1 - \frac{\tilde{K}}{\sqrt{2}}\right)$$

$\sigma_2^2 \gg \sigma_1^2 \Rightarrow$

$$\tilde{K} \approx \frac{0.354\left(-\sigma_2^2 + \sqrt{6\sigma_1^2\sigma_2^2 + \sigma_2^4}\right)/\sigma_2^2}{\sqrt{6\sigma_1^2\sigma_2^2 + \sigma_2^4}\,/\sigma_2^2}$$

$$= \frac{0.354\left(-1+\sqrt{\frac{6\sigma_1^2}{\sigma_2^2}+1}\right)}{\sqrt{\frac{6\sigma_1^2}{\sigma_2^2}+1}} \rightarrow 0 \text{ as } \sigma_2 \rightarrow \infty$$

Thus $K \rightarrow 0$ and the pole $\lambda \rightarrow a$, i.e. the same as the pole of the plant model.

When $\sigma_2 \rightarrow \infty$ the measurement $y(k)$ becomes useless and the best estimate is simply an open loop simulation of the model.

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2)
$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + e(t)$$

The stochastic process w can be modelled as the output of a LTI system G having white noise as input (Week 6):

$$\xrightarrow{v(t)} \boxed{G(s)} \xrightarrow{w(t)} \quad \Rightarrow \Phi_w(\omega) = G(j\omega)\Phi_v(\omega)G(-j\omega)$$

v white noise with unit intensity $\Rightarrow \Phi_v = 1$

$$\Phi_w(\omega) = \frac{1}{(4+\omega^2)(1+\omega^2)} = \frac{1}{(2+j\omega)(1+j\omega)} \cdot \frac{1}{(2-j\omega)(1-j\omega)}$$

$$\Rightarrow G(s) = \frac{1}{(2+s)(1+s)} = \frac{1}{2+3s+s^2}$$

$$W(s) = G(s)V(s)$$

$$s^2 W(s) + 3sW(s) + 2W(s) = V(s)$$

$$\ddot{w} + 3\dot{w} + 2w = v$$

Let $x_3 = w$ and $x_4 = \dot{w}$

$$\Rightarrow \quad \dot{x}_3 = x_4$$
$$\dot{x}_4 = -3x_4 - 2x_3 + v$$

Add these states to the original model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ x_4 \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{B_e} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{N_e} v$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_e + e(t)$$

Matrices needed: A_e, B_e, C_e, N_e, $R_1 = 1$, $R_2 = 1$

3 a) Let $z = V^{-1}x \Leftrightarrow x = Vz$

$$\dot{z} = V^{-1}\dot{x} = V^{-1}AVz + V^{-1}Bu$$

$$y = Cx = CVz$$

$$\Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0.7 & 0.7 \\ 1 & 0 & 0 \\ 0 & -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0.7 & 0 & -0.7 \\ 0.7 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0.7 & 0.7 \\ 1 & 0 & 0 \\ 0 & -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0.7 & 0.7 \\ 1 & 0 \\ -0.7 & 0.7 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0.7 & 0 & -0.7 \\ 0.7 & 0 & 0.7 \end{bmatrix} z$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1.4 & 0 & 0 \end{bmatrix} z$$

The state z_3 does not affect z_1 and z_2 and it cannot be seen in y. This is the unobservable state.

Thus $z_3 = -0.701 x_2 + 0.701 x_3 \left(= \int_0^t -0.7u_1 + 0.7u_2 \, dt \right)$

cannot be observed

3b) The least number of states needed is the number of states that are both observable and reachable. Removing z_3 gives us

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}}_{A_z} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0.7 & 0.7 \\ 1 & 0 \end{bmatrix}}_{B_z} u$$

$$y = \begin{bmatrix} 0 & 1 \\ 1.4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

The controllability matrix of this system is

$$W_r = [B_z \ \ A_z B_z] = \begin{bmatrix} \underbrace{\begin{matrix} 0.7 & 0.7 \\ 1 & 0 \end{matrix}}_{\text{linearly independent}} & \begin{matrix} x & x \\ x & x \end{matrix} \end{bmatrix}$$

$\Rightarrow \operatorname{rank}(W_r) = 2$

$\therefore$ Reachable

Thus the required number of states is 2.

Buffer tank

4a) Volume balance over tank

$$\frac{d}{dt} Ah = q_{in} - q_{out}$$

$$k_a a \sqrt{h_0 - h} - q_{out}$$

$$a = k_u \int_0^t u(\tau)\, d\tau$$

$$\frac{da}{dt} = k_u u$$

$\therefore$ A state space model is

$$\dot{h} = \frac{k_a a}{A} \sqrt{h_0 - h} - \frac{1}{A} q_{out} = f_1$$

$$\dot{a} = k_u u = f_2$$

b) Steady state o.p. $h = \bar{h} = 1$ & $\bar{q}_{out} = 0.1$

$$0 = \frac{k_a \bar{a}}{A} \sqrt{h_0 - \bar{h}} - \frac{1}{A} \bar{q}_{out}$$

$$= \bar{a} - 0.1 \Rightarrow \bar{a} = 0.1$$

Linearized model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \partial f_1 / \partial h & \partial f_1 / \partial a \\ \partial f_2 / \partial h & \partial f_2 / \partial a \end{bmatrix}_{(\bar{a}, \bar{h})}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \partial f_1 / \partial u \\ \partial f_2 / \partial u \end{bmatrix}_{(\bar{h}, \bar{a})}}_{B} u$$

$$y = \Delta h = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

Buffer tank

$$\left.\frac{\partial f_1}{\partial h}\right|_{(\bar{a},\bar{h})} = -\frac{\bar{a}k_a}{2A\sqrt{h_0-\bar{h}}} = -0.05 \qquad \frac{\partial f_1}{\partial u} = 0$$

$$\left.\frac{\partial f_1}{\partial a}\right|_{(\bar{a},\bar{h})} = \frac{k_a}{A}\sqrt{h_0-\bar{h}} = 1 \qquad \frac{\partial f_2}{\partial u} = k_u = 1$$

$$\frac{\partial f_2}{\partial h} = 0 \quad , \quad \frac{\partial f_2}{\partial a} = 0$$

$$\therefore \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.05 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x(t)$$

c) Zero order hold (piecewise constant control) discretization gives

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = C x(k)$$

where (p 34)

$$\Phi = e^{Ah} \quad \text{and} \quad \Gamma = \int_0^h e^{As}\, ds\, B$$

Determination of Φ by Laplacemethod

$$(sI-A)^{-1} = \begin{bmatrix} s+0.05 & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{1}{s(s+0.05)} \begin{bmatrix} s & 1 \\ 0 & s+0.05 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+0.05} & \frac{1}{s(s+0.05)} \\ 0 & \frac{1}{s} \end{bmatrix}$$

Buffer tank

$$\Phi = e^{Ah} = \mathcal{L}^{-1}(sI-A)^{-1}_{t=h}$$

$$= \begin{bmatrix} e^{-0.05h} & 20(1-e^{-0.05h}) \\ 0 & 1 \end{bmatrix}$$

$h = 0.1$

$$\Rightarrow \quad \Phi = \begin{bmatrix} 0.995 & 0.1 \\ 0 & 1 \end{bmatrix} \approx$$

$$\Gamma = \int_0^h e^{As} B\,ds = \int_0^h \begin{bmatrix} 20(& e^{-0.05s}) \\ & 1 \end{bmatrix} ds$$

$$= \left[\begin{bmatrix} 20(s + 20e^{-0.05s}) \\ s \end{bmatrix} \right]_0^h = \begin{bmatrix} 20h + 400(e^{-0.05h} - 1) \\ 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

$$\therefore \quad x(k+1) = \begin{bmatrix} 0.995 & 0.1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\, x(k)$$

d) (1) is reachable iff the controllability matrix has full rank

$$W_c = [\Gamma \quad \Phi\Gamma] = \begin{bmatrix} 0.005 & 0.015 \\ 0.1 & 0.1 \end{bmatrix}$$

The columns/rows are clearly linearly dependent $\Rightarrow$ full rank $\Rightarrow$ reachable

Buffer tank

4e)

$$H(z) = C[zI - \Phi]^{-1}\Gamma$$

$$= [1 \quad 0]\begin{bmatrix} z - 0.995 & -0.1 \\ 0 & z-1 \end{bmatrix}^{-1}\begin{bmatrix} 0.005 \\ \end{bmatrix}$$

$$= \frac{1}{(z \quad)(z-0.995)}[1 \quad 0]\begin{bmatrix} z-1 & 0.1 \\ 0 & z-0.995 \end{bmatrix}\begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

$$= \frac{1}{(z-1)(z-0.995)}\begin{pmatrix} 0.005z - 0.005 & 0.01 \end{pmatrix}$$

$$= \frac{0.005(z+1)}{(z-1)(z-0.995)}$$

f) Observability matrix

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.995 & 0.1 \end{bmatrix}$$ which has full rank $\Rightarrow$ both x_1 (h) and x_2 (a) can be estimated

Buffer tank

4g) Minimizing

$$V = Var\{h\} + Var(a\} + Var\{u\}$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{0}^{N-1} x_1^2(k) + x_2^2(k) + u^2(k)$$

Is the same as minimizing

$$\sum [x_1(k) \;\; x_2(k)] \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q_1} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k) \cdot \underbrace{1}_{Q_2} \cdot u(k)$$

The optimal feedback is given by the stationary Ricatti equations

$$S = \Phi^T S \Phi + Q_1 - (\Phi^T S \Gamma + Q_{12}) L$$

$$L = (Q_2 + \Gamma^T S \Gamma)^{-1} (\Gamma^T S \Phi + Q_{12}^T)$$

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} > 0$$

where, $Q_{12} = [0 \;\; 0]^T$

h) Increasing the weight on level variations (x_1) will result in reduced variations

Try e.g.

$$Q_1 = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

DC motor

4 a) Voltage balance

$$u = \frac{di}{dt} \quad Ri + V_{emf}$$

$$= L\frac{di}{dt} + Ri \quad K_m \omega$$

Torque balance

$$\tau = J\frac{d\omega}{dt} + \tau_f$$

$$K_\tau i = J\frac{d\omega}{dt} + B\sqrt{\omega}$$

$$\therefore \quad \frac{di}{dt} = -\frac{R}{L} i - \frac{K_m}{L}\omega + \frac{1}{L}u$$

$$\frac{d\omega}{dt} = \frac{K_\tau}{J} i - \frac{B}{J}\sqrt{\omega}$$

b) $K_m = K_\tau = B = J = L = 1$ & $R = 2$ $\Rightarrow$

$$\frac{di}{dt} = -2i - \omega + u = f_1$$

$$\frac{d\omega}{dt} = i - \sqrt{\omega} = f_2$$

Linearized model $x_1 = i - \bar{i}$, $x_2 = \omega - \omega$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \partial f_1/\partial i & \partial f_1/\partial \omega \\ \partial f_2/\partial i & \partial f_2/\partial \omega \end{bmatrix}_{\bar{i},\bar{\omega},\bar{u}}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix}_{\bar{i},\bar{\omega},\bar{u}}}_{B} u$$

$$\frac{\partial f_1}{\partial i} = -2, \; \frac{\partial f_1}{\partial \omega} = -1, \; \frac{\partial f_2}{\partial i} = 1, \; \frac{\partial f_2}{\partial \omega} = \frac{}{2\sqrt{\omega}}, \; \frac{\partial f_1}{\partial u} = 1, \; \frac{\partial f_2}{\partial u} = 0$$

DC motor

4b) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} -2 & -1 \\ 1 & 5 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{B} u$$

$$y = \Delta i = \underbrace{[1 \quad 0]}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c) zero order hold discretization gives

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = C x(k)$$

where (p. 35)

$$\Phi = I + A\Psi \quad \& \quad \Gamma = \Psi B$$

$$\Psi = Ih + \frac{Ah^2}{2!} + \frac{A^2h^3}{3!} + \ldots$$

$$\Psi = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 1 & -0.5 \end{bmatrix} \frac{0.01}{2} + \begin{bmatrix} 3 & 2.5 \\ -2.5 & -0.75 \end{bmatrix} \frac{0.001}{6}$$

$$= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} + \begin{bmatrix} -0.01 & -0.005 \\ 0.005 & -0.0025 \end{bmatrix} + \begin{bmatrix} 0.00050 & 0.00042 \\ -0.00042 & -0.00012 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0905 & -0.0046 \\ 0046 & 0.0974 \end{bmatrix}$$

$$\Rightarrow \Phi = \begin{bmatrix} 0.814 & -0.088 \\ 0.088 & 0.947 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0.0905 \\ 0.0046 \end{bmatrix}$$

Note: The solution $\Phi = \mathcal{L}^{-1}\{(sI-A)^{-1}\}_{t=h}$ etc at the end

DC motor

d) Controllability matrix

$$W_c = [\Gamma \;\; \Phi\Gamma] = \begin{bmatrix} 0.0905 & 0.0733 \\ 0.0046 & 0.0123 \end{bmatrix}$$

$\det(W_c) \neq 0 \Rightarrow 2$ linearly independent rows / colums
$\Rightarrow$ full rank
$\Rightarrow$ reachable

e) $H(z) = C[zI - \Phi]^{-1}\Gamma$

$$= \begin{bmatrix} z-0.814 & 0.088 \\ -0.088 & z-0.95 \end{bmatrix}^{-1} \Gamma$$

$$= \frac{1}{(z-0.814)(z-0.95)+0.088^2} \begin{bmatrix} z-0.95 & -0.088 \end{bmatrix} \begin{bmatrix} 0.0905 \\ 0.0046 \end{bmatrix}$$

$$= \frac{0.0905z - 0.0861}{z^2 - 1.76z + 0.78}$$

f) Obs. matrix $W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.814 & -0.088 \end{bmatrix}$

which has full rank $\Rightarrow x_2 = \Delta\omega$ can be estimated

g) se g) for butterlank

e) increasing weight on angular rate ω should decrease the variations in motor speed
Try for example $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$

DC motor (solution 2)

4 b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -1 \\ 1 & -0.5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$y = \Delta i = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

c) Zero order hold discretization gives

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = C x(k)$$

where [redacted]

$$\Phi = e^{Ah} \quad \text{and} \quad \Gamma = \int_0^h e^{A\tau} B \, d\tau$$

$$\Phi = \mathcal{L}^{-1}\left\{ [sI - A]^{-1} \right\}_{t=h}$$

$$[sI - A]^{-1} = \begin{bmatrix} s+2 & 1 \\ -1 & s+0.5 \end{bmatrix}^{-1} = \frac{1}{(s+2)(s+0.5)+1} \begin{bmatrix} s+0.5 & -1 \\ 1 & s+2 \end{bmatrix}$$

$$(s+2)(s+0.5)+1 = s^2 + 2.5s + 2 = 0$$

$$\Rightarrow s = -1.25 \pm i\, 0.66$$

$$\mathcal{L}^{-1}\left\{ [sI - A]^{-1} \right\}_{t=h} = \begin{bmatrix} L_1(h) + 0.5 L_2(h) & -L_2(h) \\ L_2(h) & L_1(h) + 2 L_2(h) \end{bmatrix}$$

where

$$L_1(t) = \mathcal{L}^{-1}\left\{ \frac{s}{(s+a)(s+b)} \right\} = \frac{a e^{-at} - b e^{-bt}}{a-b}$$

$$L_2(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s+a)(s+b)} \right\} = \frac{-e^{-at} + e^{-bt}}{a-b}$$

$$a = 1.25 + 0.66 i$$

$$b = 1.25 - 0.66 i$$

DC motor (solution 2)

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+a)(s+b)}\right\} = \frac{ae^{-at} - be^{-bt}}{a-b} = L_2$$

where $-a$ and $-b$ are the solutions of

$$s^2 + 2.5s + 2 = 0$$

$$(s+1.25)^2 = -2 + 1.25^2 = 0.4375$$

$$\Rightarrow \quad -a = -1.25 + i\cdot 0.66 \;,\; -b = \ldots$$

and $e^{c+id} = e^c e^{id} = e^c(\cos d + i \sin d)$

$$\sqrt{1-\zeta^2} = \sqrt{1 - 1.25^2/2} = 0.468$$

$$\omega_0\sqrt{1-\zeta^2} = 0.661$$

$$L_1(h) = \frac{1}{0.661} e^{-25\cdot 0.1} \sin(0.661\cdot 0.1) = 0.0882$$

$$\varphi(h) = \operatorname{atan}\frac{0.468}{0.884} = 0.4867$$

$$L_2(h) = \frac{}{0.468} e^{-0.125} \sin(0.0661 + 0.4867) = 0.7006$$

$$\mathcal{L}^{-1}\{[sI - A]^{-1}\} = \begin{bmatrix} L_2 + 0.5L_1 & -L_1 \\ L_1 & L_2 + 2L_1 \end{bmatrix} = \begin{bmatrix} & -0.0882 \\ 0.0882 & \end{bmatrix}$$

$L_2 = 1.58$

0.77

DC motor (solution 2)

$$a-b = 2\cdot 0.6614 i = 1.32 i$$

$$e^{-at} = e^{-(1.25+0.66i)t}$$

$$= e^{-1.25t}(\cos 0.66t - i \sin 0.66t)$$

$$e^{-ah} = 0.8806 - i 0.0583$$

$$e^{-bh} \quad 0.8806 \quad {}^{-}0.0583$$

$$ae^{-ah} - be^{-bh} = (1.25+0.66i)(0.88-0.058i) - (1.25-0.66i)(0.88+0.058i)$$

$$= -2\cdot 1.25\cdot 0.058 i + 2\cdot 0.66\cdot 0.88 i$$

$$= 1.02 i$$

$$\Rightarrow L_1(h) = \frac{1.02 i}{1.32 i} = 0.77$$

$$-e^{-ah} + e^{-bh} = 2\cdot 0.0583 = 0.1167 i$$

$$\Rightarrow L_2(h) = \frac{0.1167 i}{1.32 i} = 0.0882$$

$$\therefore \quad \Phi = \begin{bmatrix} 0.77 + 0.5\cdot 0.0882 & -0.0882 \\ 0.0882 & 0.77 + 2\cdot 0.882 \end{bmatrix}$$

$$= \begin{bmatrix} 0.814 & -0.0882 \\ 0.0882 & 0.946 \end{bmatrix}$$

$$e^{A\tau}B = \begin{bmatrix} L_1(\tau) + 0.5 L_2(\tau) & \times \\ L_2(\tau) & \times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{1.32 i} \begin{bmatrix} ae^{-a\tau} - be^{-b\tau} + 0.5(-e^{-a\tau} + e^{-b\tau}) \\ -e^{-a\tau} \quad e^{-b\tau} \end{bmatrix}$$

DC motor (solution 2)

$$\frac{1}{1.32i}\begin{bmatrix} (1.25+0.66i)e^{-1.25\tau}(\cos 0.66\tau - i\sin 0.66\tau) - e^{-1.25t}(\cos 0.66\tau - i\sin 0.66\tau) \\ \quad -(1.25-0.66i)e^{-1.25\tau}(\cos 0.66\tau + i\sin 0.66\tau) \\ \quad + e^{-1.25t}(\cos 0.66\tau + i\sin 0.66\tau) \\ 0.5\left(-e^{-1.25\tau}(\cos 0.66\tau - i\sin 0.66\tau) + e^{-1.25\tau}(\cos 0.66\tau + i\sin 0.66\tau)\right) \end{bmatrix}$$

$$= \frac{e^{-1.25\tau}}{1.32i}\begin{bmatrix} 2\cdot 1.25\cdot i\sin 0.66\tau + 2\cdot 0.66\cdot i\cdot\cos 0.66\tau \quad i\sin 0.66\tau \\ 2i\sin 0.66\tau \end{bmatrix}$$

$$= \frac{e^{-1.25\tau}}{1.32}\begin{bmatrix} 3.5\sin 0.66\tau + 1.32\cos 0.66\tau \\ 2\sin 0.66\tau \end{bmatrix}$$

$$\Gamma = \int_0^h e^{A\tau} B\,d\tau = \begin{bmatrix} \int_0^{0.1} e^{-1.25\tau}(2.65\sin 0.66\tau + \cos 0.66\tau)\,d\tau \\ 1.51\int_0^{0.1} \sin 0.66\tau\,d\tau \end{bmatrix}$$

$$= \text{---} = \begin{bmatrix} 0.0905 \\ 0.0046 \end{bmatrix}$$