Chalmers University of Technology
Department of Signals and Systems
Automatic Control Group

Exam questions for Linear Control System Design, SSY285

August 25^{th}, 2017

Cover page

1. Timeframe: 4 hours.
2. Examiner: Balazs Kulcsar, (☎1785, `kulcsar@chalmers.se`)
3. Necessary condition to obtain the exam grade is to have the course's mandatory project (all assignments and the lab) approved. Without approved mandatory project, the archived exam results are invalid.
4. 20 points can be reached in total (with 0.5 point resolution), Lic/PhD students have to reach at least 12 points to pass. For Msc students Table 1 shows the grading system.

Table 1: Grading for Msc students

Points	Grade
10 . . . 11.5	3
12 . . . 15.5	4
16 . . . 20	5

5. During this written exam, it is *optionally* permitted to use printed materials such as:
   - *Either* of the course textbooks (only 1 book): (hardcopy or plain printed version, without notes inside!)
     (i) *Feedback systems, an introduction for scientists and engineers* by K. J. Åström and R. M. Murray, ISBN-13: 978-0-691-13576-2, OR (ii) *Reglerteknikens grunder* by Bengt Lennartson, ISBN: 91-44-02416-9, (iii) *Reglerteknik : grundläggande teori*, T Glad, L Ljung.
   - 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheet can not be used.
   - Pocket calculator (non-programmable, cleared memory, without graphical plotting function).
   - Mathematical handbook Beta (without notes inside!).
6. Note that phones, tablets, computers, any other communication devices are not allowed to use during the exam session. In scheduled exam session for the course at Chalmers, teacher(s) will show up in person in the first and last 60 min.
7. Examination results will be advertised no later than 1.5 week following the examination date (via ping-pong.chalmers.se). Date and place for grade inspection will be announced (pingpong).

Good luck!

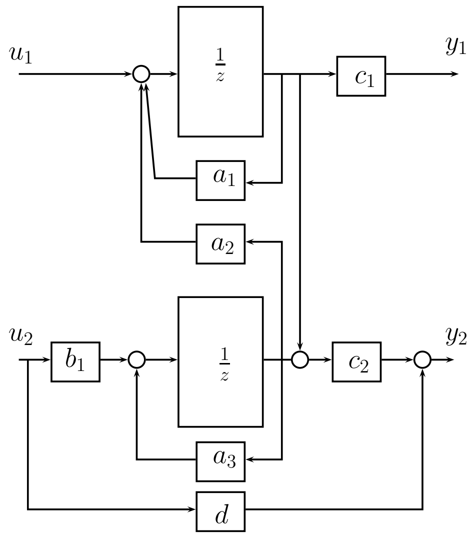

Figure 1: Open-loop block diagram

Questions

1. Briefly answer the questions with motivation (each **0.5 point**, total **2 point**).

    a) What is the state-transition matrix (continuous time, LTI state space forms)? What do we use it for?
    $\phi(t, 0) = e^{At}$, mainly we use it to create the analytic solution to the state equation. It may be used to decide controllability, observability.

    b) Your task is to diagonalize a state space model. What methodology do you recommend to use?
    If poles' multiplicity is 1, then eigenvector based similarity state transformation is suggested. If the multiplicity is larger than one, Jordan form is required with generalized eigenvalue criteria

    c) What do they stand for LQR and LQG techniques? Explain briefly how these methodology work, point out the main methodological difference.
    LQG is an output feedback controller where dynamics is triggered by the dynamic state reconstruction, by the optimal Kalman state reconstruction. Due to the state reconstruction error, it works different from a pure LQR, an optimal state feedback control law. Cost functionals are quadratic ones.

    d) What is the main conceptual difference in between the additive and multiplicative model uncertainty for LTI systems?
    The additive is an absolute metric and the multiplicative is a relative metric to describe model mismatch + math definitions, block diagrams.

2. Given the following system representation by means of block diagram in Figure 1, where $a_1 = 0.1, a_2 = 0.5, a_3 = 0.1, b_1 = 0.1$ $c_1 = 0.2, c_1 = 0.5, d = 1$.

    a) Derive the *discrete-time* state-space representation in terms of matrix difference equation (A, B, C, D) for the depicted system in Fig. 1 (**1 point**).

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0.1 & 0.5 \\ 0 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 0.2 & 0 \\ 0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k)
\end{aligned}
$$

b) By using the coefficient matrices (A, B, C, D), compute the discrete time transfer function matrix $G(z)$ (**2 point**).

From state space to transfer function matrix,

$$G(z) = C(zI_3 - A)^{-1}B + D = \begin{bmatrix} \frac{2}{10z-1} & \frac{1}{(10z-1)^2} \\ \frac{5}{10z-1} & \frac{5}{2(10z-1)^2} + \frac{1}{2(10z-1)+1} \end{bmatrix}$$

$$det(G(z)) = \frac{1 + 2(10z-1)}{(10z-1)^2}$$

c) Find the poles and the zeros of $G(z)$. Based on $G(z)$, is this a non-minimum phase representation? Is the system input-output stable? Motivate your answer (**2 point**)!

From about the denominator and numerator of the $det(G(z))$ we can find the poles: $p = 0.1$ within the stability region of the unit circle, so the system model is asymptotically IO stable, and the zero: $z = 0.05$ (within the unit circle even if with positive real value). This system model with rational transfer function is of minimum-phase since all its zeros are also inside the unit circle.

d) Is the system internally stable? Motivate your answer. (**1 point**)!

Yes, the eigenvalues of the matrix A are $\lambda_{1,2} = 0.1$

e) What is the steady state value for $y_2(k)$ while $k \mapsto \infty$ if $u_1(k) = 1$ ($U_1(z) = \frac{1}{z-1}$) and $u_2(k) = 0$ $\forall k > 0$ is applied? (subscripts refer to the input-output channel number) (**1 point**)

Since the system model is stable, FVT can be used $y_2(\infty) = \lim_{z \mapsto 1}(z-1)\frac{5}{10z-1}\frac{1}{z-1} = \frac{5}{9}$.

3. Given the following state-space representation by,

$$x(k+1) = \begin{bmatrix} -1.5 & -0.5 \\ -\alpha & 0 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{\alpha} & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u(k) \\ d(k) \end{bmatrix}$$
$$y(k) = \begin{bmatrix} 2 & \frac{1}{\alpha} \end{bmatrix} x(k)$$

with $0 < |\alpha| < \infty$, where $u(k), d(k), x(k), d(k)$ are control input, disturbance input, state vector and measured output, respectively.

a) Is the state-space representation asymptotically stable for all values of α? (**1 point**)

Not, the eigenvalues for A reads as

$$det(\lambda I_2 - A) = \lambda^2 + 1.5\lambda - \frac{\alpha}{2} = 0, \quad \lambda_{1,2} = \frac{-1.5 \pm \sqrt{1.5^2 + 2\alpha}}{2}$$

with $\alpha > 0$ we get unstable eigenvalues, but with negative α, it becomes only oscillatory but unstable (unit circle).

b) Is the state-space representation observable for all α? (**1 point**)

It is observable, except $\alpha = 2$.

c) With $\alpha = -1$ is the representation reachable and controllable? (**1 point**)

with $\alpha = -1$ create $\mathcal{R}$, it is not reachable neither controllable!

$$\mathcal{R} = \begin{bmatrix} \frac{1}{\alpha} & -\frac{3}{2\alpha} - 1 \\ 2 & -1 \end{bmatrix} \Rightarrow det(\mathcal{R}) = 0 \Rightarrow \text{ not reachable if } \alpha = -1$$

$$rank\,\mathcal{R} = 1 \neq rank\begin{bmatrix} \mathcal{R} & A^2 \end{bmatrix} = rank\begin{bmatrix} -1 & \frac{1}{2} & \frac{7}{4} & \frac{3}{4} \\ 2 & -1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} = 2 \Rightarrow \text{not controllable at } \alpha = -1$$

d) With $\alpha = 1$, find the (similarity) state transformation matrix T that renders the state space description into reachable canonical form. With the help of T transform the system into the new form, $(\tilde{A}, \tilde{B}, \tilde{C})$! (**2 point**)

Find the coefficients of the characteristics polynomial and create the similarity transformation from the reachability matrices,

$$T = \begin{bmatrix} 0.5 & 0.25 \\ -0.5 & 0.25 \end{bmatrix}, \ \tilde{A} = TAT^{-1} = \begin{bmatrix} -1.5 & 0.5 \\ 1 & 0 \end{bmatrix} \ \tilde{B} = TB_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \tilde{C} = CT^{-1} = \begin{bmatrix} 4 & 0 \end{bmatrix}$$

4. Consider the following control problem given by,

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t) \\ \dot{x}_2(t) &= x_2(t) \\ J(u) &= \frac{1}{2} \int_0^\infty \left(q_u \cdot u^2(t) + x_1^2(t) + x_2^2(t) \right) dt \end{aligned}$$

a) With $q_u = 0.1$ find the steady-state LQR state feedback gain $(\tilde{K})$ that minimizes $J(u)$ by applying the following solution matrix structure,

$$\bar{P} = \begin{bmatrix} p_1 & p_1 \\ p_1 & p_2 \end{bmatrix}$$

What are the closed loop poles? (**2 point**).

The ssystem model is not controllable. The non-controllable part of the dynamics is unstable. To solve the CARE we need that condition to be fulfilled, hence no stabilizing and optimal solution exists. Points can be also gotten by showing the CARE do not have solution in the requested structure.

b) Find the optimal cost value for the closed loop system if $x_0 = [1 \ \ 1]$? (**1 point**)

No optimal cost value can be computed, because the CARE do not have a solution.

5. Given the closed loop system as shown in Figure 2.

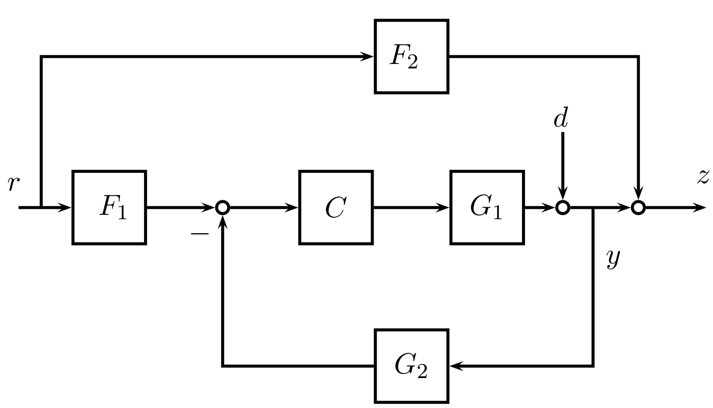

Figure 2: Closed-loop block diagram

with the plant transfer functions G_1, G_2 controller C, and pre-filter F_1,

$$G_1 = \frac{(s+1)}{(s+2)(s+3)}, G_2 = \frac{1}{(s+1)}, \quad C = 3, \quad F_1 = \frac{s+2}{s+1}.$$

a) Find the nominal transfer function matrices $G(s)$ (with unkown F_2) where $\begin{bmatrix} z \\ y \end{bmatrix} = G(s) \begin{bmatrix} r \\ d \end{bmatrix}$ (**1 point**).

$G(s) = \begin{bmatrix} F_2 + \frac{F_1CG_1}{1+CG_1G_2} & \frac{1}{CG_1G_2} \\ \frac{F_1CG_1}{1+CG_1G_2} & \frac{1}{CG_1G_2} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$, $G_{11}(s) = F_2 + \frac{3(s+2)}{s^2+5s+9}$, $G_{12} = G_{22} = \frac{1}{s^2+5s+9}$, $G_{21} = \frac{3(s+2)}{s^2+5s+9}$

b) Set F_2 such that in steady-state the transfer from r to z is 1, i.e. $r_\infty = z_\infty$ (hint: step input) (**1 point**)?

By means of FVT, $F_2 = \frac{1}{3}$

6. Given the continuous-time stochastic differential equations by,

$$\dot{x}_1(t) = -2x_1(t) + x_2(t) + v_1(t)$$
$$\dot{x}_2(t) = -3x_2(t) + v_2(t)$$

where v_1 and v_2 are elements of a vector valued *correlated*, zero mean and Gaussian white noise vector process with constant intensity matrix $V = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix}$. Find the steady-state state covariance matrix (while $t \mapsto \infty$, $\bar{P} = E\{(x(t) - m_x)(x(t) - m_x)^T\}$). (**1 point**)

Given A is stable, with $A\bar{P} + \bar{P}A + BVB^T = 0$

$$\bar{P} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$