Chalmers University of Technology
Department of Signals and Systems
Automatic Control Group

Exam questions for Linear Control System Design, SSY285

April 10^{th}, 2017

Cover page

1. Timeframe: 4 hours.
2. Examiner: Balazs Kulcsar, internal phone number: 1785, `kulcsar@chalmers.se`
3. Necessary condition to obtain the exam grade is to have the course's mandatory project (all assignments and the lab) approved. Without approved mandatory project, the archived exam results are invalid.
4. 20 points can be reached in total (with 0.5 point resolution), Lic/PhD students have to reach at least 12 points to pass. For Msc students Table 1 shows the grading system.

Table 1: Grading for Msc students

Points	Grade
10 . . . 11.5	3
12 . . . 15.5	4
16 . . . 20	5

5. During this written exam, it is *optionally* permitted to use printed materials such as:
   - *Either* of the course textbooks (only 1 book): (hardcopy or plain printed version, without notes inside!)
     (*i*) *Feedback systems, an introduction for scientists and engineers* by K. J. Åström and R. M. Murray, ISBN-13: 978-0-691-13576-2, OR (*ii*) *Reglerteknikens grunder* by Bengt Lennartson, ISBN: 91-44-02416-9, (*iii*) *Reglerteknik : grundläggande teori*, T Glad, L Ljung.
   - 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheet can not be used.
   - Pocket calculator (non-programmable, cleared memory, without graphical plotting function).
   - Mathematical handbook Beta (without notes inside!).
6. Note that phones, tablets, computers, any other communication devices are not allowed to use during the exam session. In scheduled exam session for the course at Chalmers, teacher(s) will show up in person in the first and last 60 min.
7. Examination results will be advertised no later than April 20th 2017 (pingpong.chalmers.se). Inspection of results in person, April 24th 10-11 am, E-building floor 6, room 6414 (S2 Bla Rummet).

Good luck!

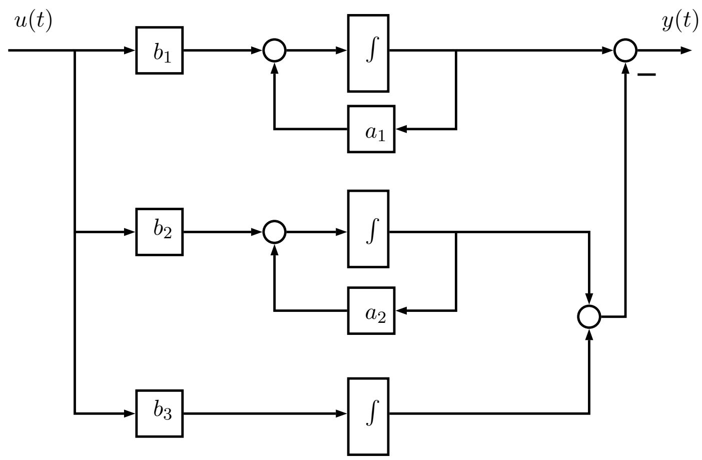

Figure 1: Open-loop state space block diagram

Questions

1. Briefly answer the questions with motivation (each **0.5 point**, total **2 point**).

   a) Explain briefly the difference between Lyapunov and asymptotic internal stability conditions for continuous time LTI state space representations.

   Real parts of the eigenvalues of the A matrix are strictly negative (asymptotic) or less then or equal to zero

   b) Briefly explain the following system properties: time-invariance, linearity.

   (i) System answer does not depend on the time when the input has been given, i.e. gives the same answer to inputs regardless when you use it, (ii) principle of weighted signals' superposition

   c) Why do we apply Loop Transfer Recovery in case of LQG controller? Explain briefly how the methodology works.

   LQG is an output feedback controller where dynamics is borrowed by the Kalman state reconstruction. Due to the state reconstruction error, it works differently from a pure LQR. We can however asymptotically recover the LQR loop behavior by increasing the process noise intensity matrix.

   d) Is the LQR controller robust? In what sense, metrics can you quantify it?

   It is robust. Both in gain margin (negative 0.5), positive (∞) and phase margin sense ± 60 degre (SISO).

2. Given the following system model by means of block diagram in Figure 1 with $a_1 = -1, a_2 = 1$, $b_1 = -1, b_2 = 2, b_3 = 0.5$.

   a) Derive the state-space representation in terms of matrix differential equation (A, B, C, D) for model depicted in Fig. 1 (**1 point**).

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} -1 \\ 2 \\ 0.5 \end{bmatrix} x(t)$$
$$y(t) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} x(t)$$

   b) By using the matrices (A, B, C, D), compute the transfer function $G(s)$ (**1 point**).

   This is a SISO diagonal system description, thus

$$G(s) = C(sI_3 - A)^{-1}B + D = \frac{-2.5s^2 + 0.5}{s^3 - s}$$

c) Based on $G(s)$, is this a non-minimum phase representation? Is the transfer function matrix strictly proper? Is the system input-output stable? Is the system internally stable? Briefly motivate your answers (**2 point**)!

$G(s) = \frac{b(s)}{a(s)} \Rightarrow b(s) = 0 \Rightarrow z = \pm 0.4472$, so the system has unstable zero and therefore it is of non minimum phase model. $p = \{1, -1, 0\}$ describes 1 stable, 1 unstable and 1 marginally stable pole and hence the system is not IO stable. Checking $eig(A)$, it returns the same eigenvalues, the conclusion is hence the same. The transfer function does have the property $G(s) = \frac{b(s)}{a(s)}, deg(a(s)) > deg(b(s))$ and hence it is proper, but strictly proper.

d) What is the steady state value for y if $u(t) = 1\ \forall t > 0$ is applied? (**0.5 point**)

Since the system is unstable, there is none, i.e. y_∞ is ∞.

e) Assume $u(t) = 0\ \forall t$ and $x(0) = x_0 = [1\ \ 1\ \ 1]^T$. What is the value for $y(1)$? (**1 point**)

We have to solve the homogenous differential equation with the

$$y(1) = Cx(1) = Ce^{At}x_0 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} e^{-1} & & \\ & e^{1} & \\ & & e^{0} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e^{-1} - e^{1} - 1$$

f) With $a_2 = 0$ is the state space representation (A, B, C, D) observable? (**1 point**)

No, since the dynamics changes in a way that x_2 and x_3 can not be distinguished anymore from about the output,

$$\mathcal{O} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ det(\mathcal{O}) = 0 \text{ develop it by the middle or last row}$$

3. Given the following state-space representation by,

$$x(k+1) = \begin{bmatrix} -1.5 & -0.5 \\ -\alpha & 0 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{\alpha} \\ 2 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 2 & \frac{1}{\alpha} \end{bmatrix} x(k)$$

with $0 < |\alpha| < \infty$,

a) Is the state-space representation asymptotically stable for all values of α? (**1 point**)

Not, the eigenvalues for A reads as

$$det(\lambda I_2 - A) = \lambda^2 + 1.5\lambda - \frac{\alpha}{2} = 0,\ \ \lambda_{1,2} = \frac{-1.5 \pm \sqrt{1.5^2 + 2\alpha}}{2}$$

with $\alpha > 0$ we get unstable eigenvalues, but with negative α, it becomes only oscillatory but stable.

b) With $\alpha = -1$ is the representation reachable and controllable? (**1 point**)

with $\alpha = -1$ create $\mathcal{R}$, not reachable not controllable!

$$\mathcal{R} = \begin{bmatrix} \frac{1}{\alpha} & -\frac{3}{2\alpha} - 1 \\ 2 & -1 \end{bmatrix} \Rightarrow det(\mathcal{R}) = 0 \Rightarrow \text{ not reachable if } \alpha = -1$$

$$rank\,\mathcal{R} = 1 \neq rank\begin{bmatrix} \mathcal{R} & A^2 \end{bmatrix} = rank\begin{bmatrix} -1 & \frac{1}{2} & \frac{7}{4} & \frac{3}{4} \\ 2 & -1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} = 2 \Rightarrow \text{not controllable at } \alpha = -1$$

c) With $\alpha = 1$, find the (similarity) state transformation matrix T that renders the state space description into diagonal form. With the help of T transform the system into a diagonal representation, $(\tilde{A}, \tilde{B}, \tilde{C})$! (**2 point**)

Find the eigenvectors and eigenvalues,

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -1.7808 & 0 \\ 0 & 0.2808 \end{bmatrix} \; \tilde{B} = TB = \begin{bmatrix} -1.5470 \\ -1.2806 \end{bmatrix}, \; \tilde{C} = CT^{-1} = \begin{bmatrix} -2.2335 & -0.4221 \end{bmatrix}$$

note, eigenvectors are not unique, in the above solution we have used orthonormal eigenvectors.

d) With $\alpha = 1$ and find $u(k) = -\tilde{K}\tilde{x}(k) + k_r r(k)$ where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$such that the closed-loop poles are allocated to -1 (both) (hint: use the diagonal form!). Find k_r such that $r_\infty = y_\infty$! (**1.5 point**)

Suppose $\tilde{B} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$$\tilde{A} - \tilde{B}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1.7808 - r_1 k_1 & 0 \\ 0 & 0.2808 - r_2 k_2 \end{bmatrix} \Rightarrow \lambda_1 - r_1 k_1 = -1 \Rightarrow k_i = \frac{\lambda_i + 1}{r_i}$$

$$k_r = \{\tilde{C}(I - \tilde{A} + \tilde{B}\tilde{K})^{-1}\tilde{B}\}^{-1}$$

4. Consider the following control problem given by,

$$\dot{x}_1(t) = -x_1(t) + u(t)$$
$$\dot{x}_2(t) = x_2(t)$$
$$J(u) = \frac{1}{2}\int_0^\infty \left(q_u \cdot u^2(t) + x_1^2(t) + x_2^2(t)\right) dt$$

a) With $q_u = 0.1$ find the steady-state LQR state feedback gain ($\tilde{K}$) that minimizes $J(u)$ by applying the following solution matrix structure,

$$\bar{P} = \begin{bmatrix} p_1 & p_1 \\ p_1 & p_2 \end{bmatrix}$$

What are the closed loop poles? (**2 point**).

The continuous time CARE returns with the following results $p_1 = 0.3162, p_2 = 1.3162$ *and the closed loop poles are* $-1, -1.3162$

b) Find the optimal cost value for the closed loop system if $x_0 = [1 \;\; 1]$? (**1 point**)

It can be computed by,

$$J^* = x_0^T \bar{P} x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0.3162 & 0.3162 \\ 0.3162 & 1.3162 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2.26$$

c) Now, suppose we do not have direct access to the states and the open loop state space description is noise corrupted as,

$$\dot{x}_1(t) = -x_1(t) + u(t) + v_1(t)$$
$$\dot{x}_2(t) = x_2(t) + v_2(t)$$
$$y(t) = x_2(t) + w(t)$$

with zero mean, uncorrelated, and normally distributed random noises w, v, intensities of the r_w is 0.1. Find the intensity matrix R_v if the solution to the steady state FARE is .

$$\bar{P} = \begin{bmatrix} 0.4494 & 0.1006 \\ 0.1006 & 0.3466 \end{bmatrix}$$

Find the poles of the closed-loop observer associated to the above solution. (**2 point**)

IBy means of the FARE, the only unknown matrix is R_v *and numerically it gives identity matrix.*

$$\bar{L} = \bar{P}C^T R_w^{-1} \Rightarrow A - \bar{L}C \Rightarrow eig(A - \bar{L}C) \Rightarrow -2.9498, \;\; -1.5161$$

5. Given the following state-space representation and cost functional by,

$$\dot{x}(t) = x(t) + 2u(t) + \sqrt{3}d(t)$$
$$y(t) = cx(t)$$
$$J(u,d) = \frac{1}{2}\int_0^{\infty} \left(y^2(\tau) + u^2(\tau)q_u - \gamma^2 d^2(t)\right) d\tau$$

where $\gamma = 1$, $q_u = \gamma$. Find the best case control input and worst case disturbance feedback gains that results in $J(u^*, d^*) = \min_u \max_d J(u,d)$. Then draw the block diagram for the closed loop (integrator, signal streams, amplifiers) (**1 point**)

$$Q_x = c^2, A = 1, B = 2, L = \sqrt{3}, Q_u = 1, \gamma = 1 \Rightarrow \text{ MCARE}$$
$$\Rightarrow 2\bar{P} + c^2 - \bar{P}^2(2 \cdot 2\frac{1}{1} - \frac{1}{1}\sqrt{3}\sqrt{3}) = 0 \Rightarrow \bar{P}^2 - 2\bar{P} - c^2 = 0 \Rightarrow \bar{P} = \frac{-(-2) + \sqrt{4 + 4c^2}}{2} > 0$$
$$\Rightarrow \bar{K}\,\bar{L}, \text{ according to the definitions}$$