Chalmers University of Technology
Department of Signals and Systems
Automatic Control Group

Exam questions for Linear Control System Design, SSY285

December 21st, 2013

Cover page

1. Timeframe: 4 hours.
2. Teacher: Balazs Kulcsar, Chalmers internal phone number: 1785
3. 20 points can be reached in total (half of a point, i.e. 0.5, can also be obtained). Note, exam grades will only be approved if all assignments and the lab have been approved by the TA too. Lic/PhD students have to reach at least 12 points to pass. For Msc students Table 1 shows the grading system.

Table 1: Grading for Msc students

Points	Grade
8 . . . 11.5	3
12 . . . 15.5	4
16 . . . 20	5

4. During this written exam, it is *optionally* permitted to optionally use printed materials such as (teacher(s) will show up and do inspection):
   - Either of the course textbooks: (hardcopy or plain printed version, without notes inside!) *Feedback systems, an introduction for scientists and engineers* by K. J. Åström and R. M. Murray, ISBN-13: 978-0-691-13576-2 or *Reglerteknikens grunder* by Bengt Lennartson, ISBN: 91-44-02416-9.
   - 1 piece of A4 paper, with *hand written* notes on both sides.
   - Pocket calculator (non-programmable, cleared memory, without graphical plotting function). Additionally to Chalmers' rules Casio fx-991 types are allowable.
   - Mathematical handbook Beta.
5. Note that phones, tablets, computers, any other communication devices are not allowed to use during the exam session. Teacher(s) will show up in person during the examination in the first and last 60 min.
6. Examination results will be advertised on January 9th 2014 (pingpong.chalmers.se). Inspection of results on January 13th 2014 9-11 am, E-building floor 5, room 5414.

Good luck and happy holidays!

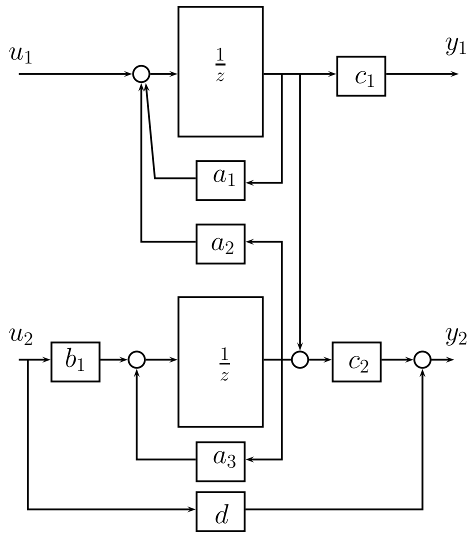

Figure 1: Open-loop block diagram

Questions

1. Given the following system representation by means of block diagram in Figure 1, where $a_1 = 0.1, a_2 = 0.5, a_3 = 0.1, b_1 = 0.1$ $c_1 = 0.2, c_1 = 0.5, d = 1$.

   a) Derive the *discrete-time* state-space representation in terms of matrix difference equation (A, B, C, D) for the depicted system in Fig. 1 (**1 point**).

   b) By using the coefficient matrices (A, B, C, D), compute the discrete time transfer function matrix $G(z)$ (**2 point**).

   c) Find the poles and the zeros of $G(z)$. Based on $G(z)$, is this a non-minimum phase representation? Is the system input-output stable? Motivate your answer (**2 point**)!

2. Given the following state space representation by,

$$\dot{x}(t) = \begin{bmatrix} -1.5 & \beta \\ -1 & \gamma \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + \sqrt{\beta \cdot \gamma}\, u(t)$$

   where β and γ are real valued finite scalars.

   a) Find value(s) of β and γ for which the state-space represnetation becomes non-minimal (**2 point**).

   b) With $\beta = \gamma = 1$ find the transformation matrix T that diagonalizes the above representation and with this T, transform the above mentioned state-space representation to a diagonal form $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$. Calculate the values $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$(**3 point**).

3. Consider the following continuous time system given by a state space representation as,

$$\dot{x}_1(t) = x_2(t) + u(t)$$
$$\dot{x}_2(t) = u(t)$$
$$J(u) = \int_0^\infty \left(x^T(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + u^2(t) \right) dt.$$

   a) Find the optimal (steady-state LQR) feedback gain $(\bar{K})$ that minimizes $J(u)$ by applying the diagonal solution structure as $\bar{P} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ (**2 point**).

   b) Compute the closed-loop optimal poles (**1 point**).

4. Consider the following time discrete problem by,

$$
\begin{aligned}
x(k+1) &= -x(k) + v_1(k) + 2v_2(k), \\
y(k) &= x(k) + w(k),
\end{aligned}
$$

where the scalar noise intensities are $r_w = 1$, $r_{v1} = 1$, $r_{v2} = 1$. All noises are normally distributed uncorrelated white noise sequences.

a) Find the optimal steady state Kalman filter gain $\bar{L}$ to minimize the "delayed" state error covariance, $\bar{p} = E\{\tilde{x}(k|k-1)\tilde{x}(k|k-1)^T\} > 0$ with structured solution matrix (**2 point**).

b)Find the closed loop system matrix $A - \bar{L}C$ with previously obtained gain $\bar{L}$. Draw the block diagram of plant and the filter together (**1 point**).

5. Given the closed loop system as shown in Figure 2.

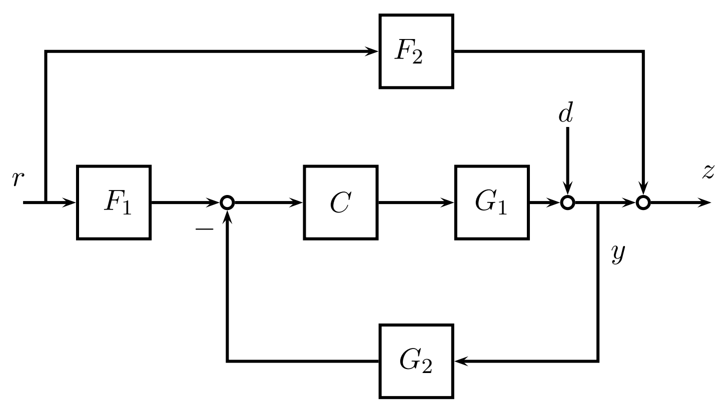

Figure 2: Closed-loop block diagram

with the plant transfer functions G_1, G_2 controller C, and pre-filter F_1,

$$
G_1 = \frac{(s+1)}{(s+2)(s+3)}, G_2 = \frac{1}{(s+1)}, \quad C = 3, \quad F_1 = \frac{s+2}{s+1}.
$$

(*a*) Find the nominal transfer function matrices $G(s)$ (with unkown F_2) where $\begin{bmatrix} z \\ y \end{bmatrix} = G \begin{bmatrix} r \\ d \end{bmatrix}$ (**1 point**).

(*b*) Set F_2 such that in steady-state the transfer from r to z is 1, i.e. $r_\infty = z_\infty$ (hint: step input) (**1 point**)?

6. *(challenging exercise) Given the continuous-time stochastic differential equations by,

$$
\begin{aligned}
\dot{x}_1(t) &= -2x_1(t) + x_2(t) + v_1(t) \\
\dot{x}_2(t) &= -3x_2(t) + v_2(t)
\end{aligned}
$$

where v_1 and v_2 are elements of a vector valued *correlated*, zero mean and Gaussian white noise vector process with constant intensity matrix V. Compute this noise intensity matrix V provided the steady-state state covariance matrix (while $t \mapsto \infty$, $\bar{P} = E\{(x(t) - m_x)(x(t) - m_x)^T\}$) is know as,

$$
\bar{P} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ (\textbf{2 point})}.
$$