CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Exam 2016-04-08

14:00 – 18:00

Teachers: Bo Egardt (tel 3721) and Faisal Altaf (tel 1774). We will visit twice during the exam.

The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 sheet (front+back page) with your own notes.
- Mathematics Handbook (Beta).

Note: Solutions should be given in **English**! They may be short, but should always be **clear, readable and well motivated**!

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5).
Review of the grading is offered on April 25 at 12.00 – 13.00 in the office of Faisal Altaf. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

GOOD LUCK!

Problem 1.

a. An active set method is used for two versions of the standard LQ type MPC studied in the course: (i) the non-condensed version and (ii) the condensed one. Disregarding any inequality constraints, what is the size of the linear system of equations to solve in each iteration, if the problem has horizons $N = M = 20$, number of states $n = 5$ and number of inputs $m = 1$? (2 p)

b. You are asked to implement an MPC controller of the type studied in the course. However, there is not time enough to think about what to do when the algorithm encounters infeasibility. What can be done to completely avoid this risk? (2 p)

c. Consider an LTI system on standard (A, B, C) state-space form. Assume that the system is square (i.e. the number of inputs and outputs are equal) and that the transfer function $H(z)$ has the property that $H(1)$ is invertible. Show that an arbitrary setpoint vector y_{sp} is a feasible (attainable) steady-state target. (2 p)

d. Explain what is meant by *recursive feasibility.* (2 p)

e. Consider a standard quadratic programming (QP) problem with both equality and inequality constraints. What is the main idea behind the barrier method to solve the problem? Is the transformed problem convex? Motivate your answer! (2 p)

Solution:

a. *There are* $n \cdot N = 100$ *state variables and* $m \cdot N = 20$ *control variables. Hence, there are 120 primal variables in the non-condensed version and 20 in the condensed. In addition, there are* $n \cdot N = 100$ *dual variables for the equality constraints in the non-condensed version. Hence, the sizes are* $120 + 100 = 220$ *and* 20*, respectively.*

b. *The only way to completely avoid the risk is to not allow state constraints (including terminal constraints).*

c. *The steady-state target should fulfil the equation*

$$\begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}$$

Define $u_s = H(1)^{-1}y_{sp}$. *It then follows that* $H(1)u_s = y_{sp}$ *or* $C(I - A)^{-1}Bu_s = y_{sp}$. *By defining* $x_s = (I - A)^{-1}Bu_s$, *we thus have*

$$
\begin{aligned}
(I - A)x_s &= Bu_s \\
Cx_s &= y_{sp}
\end{aligned}
$$

which are the required conditions for any steady state target.

d. *Recursive feasibility is the property that solving the MPC optimization problem for an initial feasible state results in the next state being feasible as well. This results in a sequence of feasible (solvable) optimal control problems.*

e. *The main idea behind the barrier method is to get rid of the inequality constraints of the form* $g_i^T x \leq h_i$ *by adding terms of the form* $-\log(h_i - g_i^T x)$ *to the objective. The idea is that these terms act like "barriers" towards entering the infeasible region. The objective stays convex, since the* $\log$ *function is concave, and hence* $-\log$ *is convex.*

Problem 2.
In this problem, we consider solving the following quadratic program using the Newton method:

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^T Q x + p^T x, \quad Q \succ 0 \\ \text{subject to} \quad & Ax = b \end{aligned}$$

a. Show how the Newton update equation for solving $r(x) = 0$ can be derived from a linear approximation of the function $r(x)$ at the current iterate. (1 p)

b. Apply the Newton method to the KKT conditions of the QP given above and give an expression for the Newton **update** of primal and dual variables. (2 p)

c. Show that the Newton step derived in (b) actually gives the solution of the KKT equations in one step. (2 p)

Solution:

a. *Let x be the current iterate ("guess") and Δx be the update step. Putting the linear approximation around x to 0 gives:*

$$r(x + \Delta x) \approx r(x) + \frac{\partial r(x)}{\partial x}\Delta x = 0 \quad \Rightarrow \quad \frac{\partial r(x)}{\partial x}\Delta x = -r(x)$$

b. *The KKT conditions for the problem are (using the Lagrangian $L(x, \nu) = \frac{1}{2}x^T Q x + p^T x + \nu^T(Ax - b)$):*

$$\begin{aligned} \nabla L(x, \nu) &= Qx + p + A^T\nu = 0 \\ h(x) &= Ax - b = 0 \end{aligned}$$

Applying the update equation in (a) to this system of equations of the variable (x, ν) gives

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \nu \end{bmatrix} = - \begin{bmatrix} Qx + p + A^T\nu \\ Ax - b \end{bmatrix}$$

c. *Using the notation $x^+ = x + \Delta x$ and similarly for ν, the update equation given in (b) can be written as*

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^+ \\ \nu^+ \end{bmatrix} = \begin{bmatrix} -p \\ b \end{bmatrix}$$

which is identical to the KKT conditions given above. The Newton method thus gives the optimal primal and dual variables in one step.

Problem 3.
Consider a first order system described by the model

$$x(k+1) = x(k) + u(k)$$

We want to construct an LQ based MPC for this system based on minimization of the 1-step ahead cost function (where as usual 'current time' k has been placed at the origin)

$$V_1(x(0), u(0)) = x^2(0) + x^2(1),$$

with state terminal constraint $|x(1)| \leq 1$ and control constraint $|u(0)| \leq 1$.

a. Begin by neglecting the constraints. Determine the control law resulting from the *unconstrained* LQ problem. What is the feasible set of initial states? (1 p)

b. Now include the constraints on u and x. What is now the feasible set? (1 p)

c. In order to enlarge the set if feasible states in (b), the terminal constraint on x is dropped (but the constraint on u is kept). Determine the control law for the constrained MPC formulation.
Hint: Use the KKT conditions. (2 p)

d. Determine the closed-loop dynamics for the latter case. (1 p)

Solution:

a. The cost function is, with $x = x(0)$ and $u = u(0)$,

$$V_1(x, u) = x^2(0) + x^2(1) = x^2 + (x+u)^2 = 2x^2 + 2xu + u^2$$

From this follows that the unconstrained control law, obtained by putting $\frac{\partial}{\partial u} V_1(x, u) = 0$, is given by

$$u = -x,$$

which is a dead-beat control law. Since the constraints have been neglected, the feasible set is the entire real line.

b. With the constraint on u, x can be changed in one step at most ± 1. From the terminal constraint $|x(1)| \leq 1$ it follows that the feasible set is given by $|x| \leq 2$.

c. *Expressing the constraints as* $u - 1 \leq 0$ *and* $-u - 1 \leq 0$, *respectively, the Lagrangian becomes*

$$\mathcal{L}(u, \lambda_1, \lambda_2) = 2x^2 + 2xu + u^2 + \lambda_1(u-1) + \lambda_2(-u-1)$$

The KKT conditions are therefore

$$\begin{gathered} 2x + 2u + \lambda_1 - \lambda_2 = 0 \\ u - 1 \leq 0 \\ -u - 1 \leq 0 \\ \lambda_1 \geq 0, \quad \lambda_2 \geq 0 \\ \lambda_1(u-1) = 0, \quad \lambda_2(-u-1) = 0 \end{gathered}$$

Investigating the different cases implied by the complementarity conditions, the following constrained MPC control law is obtained:

$$u = \begin{cases} 1, & x < -1 \\ -x, & -1 \leq x \leq 1 \\ -1, & x > 1 \end{cases}$$

d. *From (c) it follows that the closed-loop system is described by*

$$x(k+1) = \begin{cases} x(k) + 1, & x(k) < -1 \\ 0, & -1 \leq x(k) \leq 1 \\ x(k) - 1, & x(k) > 1 \end{cases}$$

i.e. the closed-loop system approaches the origin in a linear fashion until it reaches the unit "ball", and then converges to the origin in one step.

Problem 4.

Consider the following system with two inputs and two outputs:

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a. Is the output setpoint $y_{sp} = \begin{bmatrix} 2 & 2.2 \end{bmatrix}^T$ attainable, i.e. is there a steady-state target (x_s, u_s) such the $Cx_s = y_{sp}$? (2 p)

b. Now assume only the first input is available for control, i.e. $u_2 \equiv 0$. Is the output setpoint still attainable? Determine the setpoint target by minimising $\|Cx_s - y_{sp}\|$. (3 p)

Solution:

a. *Having 2 inputs and 2 outputs makes it possible to solve the conditions for setpoint tracking, i.e.*

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix}$$

From the first block row it follows that

$$x_s = \begin{bmatrix} 1 & 0 \\ 0 & 10 \\ 2 & 2 \end{bmatrix} u_s$$

which gives the steady-state output

$$y_s = \begin{bmatrix} 1 & 10 \\ 2 & 2 \end{bmatrix} u_s$$

so that with $u_s^T = \begin{bmatrix} 1 & 0.1 \end{bmatrix}$ *the setpoint is attained.*

b. *From (a) it follows that in steady-state we now have* $y_s = \begin{bmatrix} 1 & 2 \end{bmatrix}^T u_{1s}$, *with* u_1 *the remaining scalar input. Hence, the output setpoint* $y_{sp} = \begin{bmatrix} 2 & 2.2 \end{bmatrix}^T$ *can not be reached in steady-state.*
Minimising $\|Cx_s - y_{sp}\|^2 = (u_{1s} - 2)^2 + (2u_{1s} - 2.2)^2$ *by setting the derivative equal to zero gives* $3u_{1s} - 4.2 = 0$, *hence* $u_{1s} = 1.4$. *The setpoint target becomes* $x_s = \begin{bmatrix} 1.4 & 0 & 2.8 \end{bmatrix}$, $u_{1s} = 1.4$ *and thus* $y_s = \begin{bmatrix} 1.4 \; 2.8 \end{bmatrix}$.
Remark: note that $[1.4 \;\, 2.8]$ *is the orthogonal projection of the vector* $[2 \; 2.2]$ *on the straight line described by* $y_s = \begin{bmatrix} 1 & 2 \end{bmatrix}^T u_{1s}$.

Problem 5.
Consider a model predictive controller based on the following finite time optimal control problem:

$$\begin{aligned} \min_{\{u(0),u(1),\ldots,u(N-1)\}} \quad & \sum_{k=0}^{N-1} \big(x^T(k)Qx(k) + u^T(k)Ru(k)\big) + x^T(N)P_f x(N) \\ \text{subject to} \quad & x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \\ & Cx(k) + Du(k) \le f, \quad k = 0, \ldots, N-1 \end{aligned} \tag{1}$$

In this problem we have $N = 3$, and the controls $u(k)$ are re-parametrized in terms of a nominal, stabilizing feedback as

$$u(k) = Kx(k) + v(k),$$

which implies that the decision variables are now $\{v(k)\}$.

a. Denote by $\mathbf{v}$ the vector of decision variables, i.e.

$$\mathbf{v} = \begin{bmatrix} v(0) \\ v(1) \\ v(2) \end{bmatrix}$$

Find matrices H and E, vectors h and g, and a constant c such that the optimal control problem (1) can be rewritten as

$$\begin{aligned} \min_{\mathbf{v}} \quad & \mathbf{v}^T H \mathbf{v} + h^T \mathbf{v} + c \\ \text{subject to} \quad & E\mathbf{v} \le g \end{aligned} \tag{2}$$

(2 p)

b. Assume now that there are no constraints in the problem (2). Hence, at every time instant, the unconstrained problem

$$\min_{\mathbf{v}} \quad \mathbf{v}^T H \mathbf{v} + h^T \mathbf{v} + c$$

is solved, giving the optimal solution

$$\mathbf{v}^* = \begin{bmatrix} v^*(0;x) \\ v^*(1;x) \\ v^*(2;x) \end{bmatrix}.$$

Suggest a choice of the terminal cost P_f, such that the closed-loop system

$$x(k+1) = (A + BK)x(k) + Bv^*(0; x(k))$$

is stable. (3 p)

Hint 1: The "basic stability assumption" from the lecture notes reads

$$\min_{u\in\mathbb{U}} \{V_f(f(x,u)) + l(x,u) \,|\, f(x,u) \in \mathbb{X}_f\} \leq V_f(x), \quad \forall x \in \mathbb{X}_f$$

Hint 2: The Lyapunov equation in discrete time reads

$$A^T S A - S = -Q$$

Solution:

a. *With the re-parametrization, the state equation becomes*

$$x(k+1) = (A+BK)x(k) + Bv(k) = \mathcal{A}x(k) + Bv(k)$$

so that

$$\begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} \mathcal{A} \\ \mathcal{A}^2 \\ \mathcal{A}^3 \end{bmatrix} x(0) + \begin{bmatrix} B & 0 & 0 \\ \mathcal{A}B & B & 0 \\ \mathcal{A}^2 B & \mathcal{A}B & B \end{bmatrix} \begin{bmatrix} v(0) \\ v(1) \\ v(2) \end{bmatrix}$$

or, with a more compact notation,

$$\mathcal{X} = \Omega x(0) + \Gamma \mathbf{v}$$

Introducing the matrices $\bar{Q} =$ diag (Q, Q, P_f) and $\bar{R} =$ diag (R, R, R), the objective can now be written

$$\begin{aligned} x^T(0)Qx(0) + \mathcal{X}^T \bar{Q}\, \mathcal{X} + \mathbf{v}^T \bar{R}\mathbf{v} \\ = x^T(0)Qx(0) + (\Omega x(0) + \Gamma\mathbf{v})^T \bar{Q}(\Omega x(0) + \Gamma \mathbf{v}) + \mathbf{v}^T \bar{R}\mathbf{v} \\ = \mathbf{v}^T(\Gamma^T \bar{Q}\Gamma + \bar{R})\mathbf{v} + 2x^T(0)\Omega^T \bar{Q}\Gamma\mathbf{v} + x^T(0)(Q + \Omega^T \bar{Q}\Omega)x(0) \\ = \mathbf{v}^T H \mathbf{v} + h^T \mathbf{v} + c \end{aligned}$$

with obvious definitions of H, h and c. In a similar way, we get the constraint

$$\begin{bmatrix} C+DK \\ (C+DK)\mathcal{A} \\ (C+DK)\mathcal{A}^2 \end{bmatrix} x(0) + \begin{bmatrix} DB & 0 & 0 \\ (C+DK)B & DB & 0 \\ (C+DK)\mathcal{A}B & (C+DK)B & DB \end{bmatrix} \begin{bmatrix} v(0) \\ v(1) \\ v(2) \end{bmatrix} \leq \begin{bmatrix} f \\ f \\ f \end{bmatrix}$$

or

$$\begin{bmatrix} DB & 0 & 0 \\ (C+DK)B & DB & 0 \\ (C+DK)\mathcal{A}B & (C+DK)B & DB \end{bmatrix} \mathbf{v} \leq \begin{bmatrix} f \\ f \\ f \end{bmatrix} - \begin{bmatrix} C+DK \\ (C+DK)\mathcal{A} \\ (C+DK)\mathcal{A}^2 \end{bmatrix} x(0)$$

b. *First note that, since there is no terminal constraint ($\mathbb{X}_f = \mathbb{X}$), we need to find $V_f(x) = x^T P_f x$, such that the "basic stability assumption" is satisfied for all x (and with u replaced by v). Since the feedback in the re-parametrized problem is stabilizing, $\mathcal{A} = A + BK$ is a stable matrix and hence, for any $Q_s \succ 0$ there is an $S \succ 0$ such that $\mathcal{A}^T S \mathcal{A} - S = -Q_s$, or*

$$x^T(A+BK)^T S(A+BK)x + x^T Q_s x = x^T Sx, \quad \forall x$$

Choosing $Q_s = Q + K^T RK$ gives

$$x^T(A+BK)^T S(A+BK)x + x^T Qx + (Kx)^T R(Kx) = x^T Sx, \quad \forall x$$

Hence, the inequality in the "basic stability assumption" is satisfied with equality and the choice $v = 0$. The conclusion is that P_f should be chosen as the solution of the Lyapunov equation

$$(A+BK)^T P_f(A+BK) - P_f = -(Q + K^T RK)$$

THE END!