CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Exam 2016-03-18

08:30 – 12:30

Teachers: Bo Egardt (tel 3721) and Faisal Altaf (tel 1774). We will visit twice during the exam.

The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 sheet (front+back page) with your own notes.
- Mathematics Handbook (Beta).

Note: Solutions should be given in **English**! They may be short, but should always be **clear, readable and well motivated**!

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5).
Review of the grading is offered on April 5 at 12.00 – 13.00 in the office of Faisal Altaf. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

GOOD LUCK!

Problem 1.

a. The (steady-state) setpoint tracking problem for the case with more control inputs than controlled outputs presents a potential problem. *Which* is the problem, and *how* can it be avoided? (2 p)

b. The *condensed* version of the optimization problem being solved in the LQ type MPC studied during the course offers an advantage relative to the non-condensed version – which? On the other hand, it "destroys" an attractive feature of the non-condensed version – which? (2 p)

c. When proving stability of a model predictive controller, a fundamental step is the following assumption (taken from the lecture notes):

$$\min_{u \in \mathbb{U}} \{V_f(f(x,u)) + l(x,u) \mid f(x,u) \in \mathbb{X}_f\} \leq V_f(x), \quad \forall x \in \mathbb{X}_f$$

Explain how this assumption can be shown to hold by choosing $\mathbb{X}_f = \{0\}$. (2 p)

d. The MPC algorithm studied during the course is based on i) a linear model; ii) quadratic objective, and iii) affine inequality constraints. Explain in what way these assumptions make the MPC optimization problem *convex*. (2 p)

e. Consider a standard quadratic programming (QP) problem with both equality and inequality constraints. What is the main idea behind the barrier method to solve the problem? Is the transformed problem convex? Motivate your answer! (2 p)

Solution:

a. *The problem is that there usually exist multiple solutions to the setpoint tracking problem. Setpoints for inputs, in addition to setpoints for outputs, may be used to avoid this.*

b. *In the condensed version, the states are expressed in terms of the initial state and the future controls, so that only the control actions are the decision variables. In this way, fewer decision variables are obtained, but the price being paid is that the sparsity of the original problem, involving both states and controls as decision variables, is lost.*

c. *The inequality is fulfilled with equality for* $x = u = 0$ *if* $f(0,0) = 0$, $V_f(0) = 0$ *and* $l(0,0) = 0$.

d. *The **objective is convex**, since a quadratic function with positive semidefinite Hessian is convex. The linear model implies that state predictions and future controls are related via **affine equality constraints**. Affine inequality constraints on input and state constitute **convex inequality constraints**. Hence, all the conditions for a convex optimization problem on standard form are fulfilled.*

e. *The main idea behind the barrier method is to get rid of the inequality constraints of the form $g_i^T x \leq h_i$ by adding terms of the form $-\log(h_i - g_i^T x)$ to the objective. The idea is that these terms act like "barriers" towards entering the infeasible region. The objective stays convex, since the* log *function is concave, and hence* $-\log$ *is convex.*

Problem 2.
In this problem, we consider solving the following quadratic program using the Newton method:

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}x^T Q x + p^T x, \quad Q \succ 0 \\ \text{subject to} \quad & Ax = b \end{aligned}$$

a. Show how the Newton update equation for solving $r(x) = 0$ can be derived from a linear approximation of the function $r(x)$ at the current iterate. (1 p)

b. Apply the Newton method to the KKT conditions of the QP given above and give an expression for the Newton **update** of primal and dual variables. (2 p)

c. Show that the Newton step derived in (b) actually gives the solution of the KKT equations in one step. (2 p)

Solution:

a. *Let x be the current iterate ("guess") and Δx be the update step. Putting the linear approximation around x to 0 gives:*

$$r(x + \Delta x) \approx r(x) + \frac{\partial r(x)}{\partial x}\Delta x = 0 \quad \Rightarrow \quad \frac{\partial r(x)}{\partial x}\Delta x = -r(x)$$

b. *The KKT conditions for the problem are (using the Lagrangian $L(x, \nu) = \frac{1}{2}x^T Q x + p^T x + \nu^T(Ax - b)$):*

$$\begin{aligned} \nabla L(x, \nu) &= Qx + p + A^T \nu = 0 \\ h(x) &= Ax - b = 0 \end{aligned}$$

Applying the update equation in (a) to this system of equations of the variable (x, ν) gives

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \nu \end{bmatrix} = - \begin{bmatrix} Qx + p + A^T \nu \\ Ax - b \end{bmatrix}$$

c. *Using the notation $x^+ = x + \Delta x$ and similarly for ν, the update equation given in (b) can be written as*

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^+ \\ \nu^+ \end{bmatrix} = \begin{bmatrix} -p \\ b \end{bmatrix}$$

which is identical to the KKT conditions given above. The Newton method thus gives the optimal primal and dual variables in one step.

Problem 3.
Consider a first order system described by the model

$$x(k+1) = 0.5x(k) + u(k), \quad u(k) \geq 0$$

where it should be noticed that only non-negative control inputs are allowed. We want to construct an LQ based MPC for this system based on minimizing the 1-step ahead cost function

$$V_1(x(0), u(0)) = x^2(1) + u^2(0),$$

where as usual 'current time' k has been placed at the origin.

a. Determine the cost function $V_1(x, u)$ as a function of current state $x = x(0)$ and control candidate $u = u(0)$. Based on this, determine the control law resulting from the *unconstrained* LQ problem. (2 p)

b. Now assume that the control constraint is taken into consideration, i.e. we would like to minimize V_1 under the constraint

$$u \geq 0.$$

Determine the control law for the constrained MPC formulation. (2 p)

c. Show that the state converges to zero for the closed-loop system obtained with the constrained MPC. (1 p)

Solution:

a. The cost function is

$$V_1(x,u) = x^2(1) + u^2(0) = (0.5x+u)^2 + u^2 = 2(u+\frac{x}{4})^2 + \frac{1}{8}x^2$$

From this follows that the unconstrained control law, minimizing V_1, is given by

$$u = -\frac{x}{4}$$

b. With the constraint on u, the minimizing control is not any longer allowed for positive x. However, for positive x, it follows from the expression

$$V_1(x,u) = 2u^2 + xu + \frac{1}{4}x^2$$

that $V_1(x,u)$ is minimized by the choice $u = 0$ (both u-dependent terms increase with u). The constrained MPC control law is hence

$$u(x) = \begin{cases} -x/4 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

c. From (b) it follows that the closed-loop system is described by either of two equations

$$x(k+1) = \begin{cases} 0.25x(k) & x(0) < 0 \\ 0.5x(k) & x(0) \leq 0 \end{cases}$$

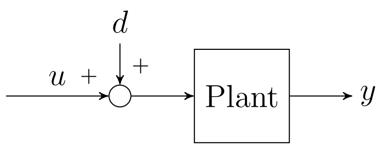
i.e the system is governed by the open-loop dynamics if the initial state is negative. In either case, the state converges to zero exponentially and the closed-loop system is stable.

Problem 4.

Consider a process described by the standard linear state space model

$$x^+ = Ax + Bu$$
$$y = Cx$$

It is assumed that the model is detectable. This model is extended with a vector d of additive disturbances at the plant input as shown below. The dimension of d is equal to the dimension of u.



a. Assuming d is modeled as constant but unknown, show how the original plant model can be augmented to include the disturbance. (1 p)

b. In the SISO case, determine a condition on the plant zeros that guarantees that the constant but unknown disturbance can be estimated based on measurements of inputs and outputs.
Hint: The test on detectability of the augmented system, given during the course, can be formulated as

$$\text{rank} \begin{bmatrix} zI - A & -B_d \\ C & C_d \end{bmatrix} = n + n_d, \text{ for } z = 1$$

(2 p)

c. For the following special case, a first order SISO system, determine a state observer with deadbeat error dynamics (all eigenvalues in the origin):

$$\begin{aligned} x(k+1) &= 0.9x(k) + 0.5\big(u(k) + d(k)\big) \\ y(k) &= x(k) \end{aligned}$$

(2 p)

Solution:

a. *The augmented model is*

$$\begin{bmatrix} x \\ d \end{bmatrix}^+ = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

b. *A necessary and sufficient condition for the augmented system to be detectable is that (the system is SISO, i.e. square, so that the rank can be tested by looking at the determinant)*

$$\begin{vmatrix} zI - A & -B \\ C & 0 \end{vmatrix} \neq 0, \text{ for } z = 1$$

Multiplying the first block row by $C(zI - A)^{-1}$ *and subtracting from the second block row will not affect the determinant:*

$$\begin{vmatrix} zI - A & -B \\ C & 0 \end{vmatrix} = \begin{vmatrix} zI - A & -B \\ 0 & C(zI - A)^{-1}B \end{vmatrix} = \det(zI - A)H(z) = n(z)$$

where $H(z) = C(zI - A)^{-1}B$ *is the transfer function of the system and* $n(z)$ *its numerator polynomial. Hence, the detectability condition is that there must be no zero of the transfer function in* $z = 1$.

c. The augmented model for the example is

$$\begin{bmatrix} x \\ d \end{bmatrix}^+ = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

A standard observer for the augmented system is given by

$$\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}^+ = \left(\begin{bmatrix} 0.9 & 0.5 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} l_x \\ l_d \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} l_x \\ l_d \end{bmatrix} y(k)$$

The characteristic polynomial for the error dynamics is given by

$$(z - (0.9 - l_x))(z - 1) + 0.5 l_d = z^2 + (l_x - 1.9)z + 0.9 - l_x + 0.5 l_d$$

which implies that the choice $l_x = 1.9$ *and* $l_d = 2$ *gives deadbeat dynamics.*

Problem 5.
In this problem we will investigate the following two alternative MPC formulations, both without constraints on the control input, for the nth order system

$$x^+ = Ax + Bu$$

I. The first formulation (familiar from the course) is based on the following optimization problem:

$$\min_{x(\cdot),u(\cdot)} \quad \sum_{k=0}^{N-1} l_s(x(k), u(k))$$
$$\text{subject to} \quad x^+ = Ax + Bu, \quad x(0) = x_0, \quad x(N) = x_s,$$

where the stage cost is given by

$$l_s(x, u) = \frac{1}{2}\big((x - x_s)^T Q (x - x_s) + (u - u_s)^T R (u - u_s)\big).$$

The steady state target (x_s, u_s) is obtained from the optimization problem

$$\min_{x_s,u_s} \quad l_{sp}(x_s, u_s)$$
$$\text{subject to} \quad \begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0$$

where setpoints x_{sp}, u_{sp} for state and input have been defined and

$$l_{sp}(x_s, u_s) = \frac{1}{2}\big((x_s - x_{sp})^T Q(x_s - x_{sp}) + (u_s - u_{sp})^T R(u_s - u_{sp})\big).$$

II. The second formulation is instead based on the following optimization problem:

$$\begin{aligned} \min_{x(\cdot),u(\cdot)} \quad & \sum_{k=0}^{N-1} l_{sp}(x(k), u(k)) \\ \text{subject to} \quad & x^+ = Ax + Bu, \quad x(0) = x_0, \quad x(N) = x_s, \end{aligned}$$

with the stage cost

$$l_{sp}(x, u) = \frac{1}{2}\big((x - x_{sp})^T Q(x - x_{sp}) + (u - u_{sp})^T R(u - u_{sp})\big). \tag{1}$$

a. Show that in formulation (II) above, the solution will be unaffected by modifying the stage cost to $l^*_{sp}(x, u) = l_{sp}(x, u) + \nu^T\big((I - A)x - Bu\big)$, where ν is an arbitrary $n-$vector and $l_{sp}(x, u)$ is given in (1). (2 p)

b. Now, let ν in (a) above be the optimal dual variable when solving the steady state target problem in formulation (I). Use the result in (a) to show that formulations (I) and (II) will give the same solutions.
Hint: Use the KKT conditions to characterize the solution to the steady state target problem. (3 p)

Solution:

a. *From the model constraint follows that* $(I - A)x(k) - Bu(k) = x(k) - x(k + 1)$. *Hence,*

$$\begin{aligned} \sum_{k=0}^{N-1} \nu^T\big((I - A)x(k) - Bu(k)\big) &= \sum_{k=0}^{N-1} \nu^T\big(x(k) - x(k + 1)\big) \\ &= \nu^T\big(x(0) - x(N)\big) = \nu^T\big(x_0 - x_s\big), \end{aligned}$$

which does not depend on the optimization variables and therefore does not affect the solution.

b. *The solution to the steady state target problem satisfies the KKT conditions, in particular that the gradient of the Lagrangian* $\mathcal{L} = l_{sp}(x_s, u_s) + \nu^T\big((I-A)x_s - Bu_s\big)$ *is equal to zero, giving (differentiate w.r.t.* x_s *and* u_s*)*

$$(x_s - x_{sp})^T Q + \nu^T(I - A) = 0, \qquad (u_s - u_{sp})^T R - \nu^T B = 0$$

Multiplying from the right by x *and* u*, respectively, and using the result in (a) implies that adding the term*

$$-\big((x_s - x_{sp})^T Qx + (u_s - u_{sp})^T Ru\big)$$

from the stage cost for (II) will not affect the solution. However, this modification to the stage cost of (II) will result in the stage cost of (I), modulo constant terms. Hence, formulations (I) and (II) will give the same solution.

THE END!