CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Exam 2015-04-17

14:00 – 18:00

Teacher: Bo Egardt, tel 3721.

The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 sheet (front+back page) with your own notes.
- Mathematics Handbook (Beta).

Note: Solutions should be given in **English**! They may be short, but should always be **clear, readable and well motivated**!

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5).
Review of the grading is offered on April 30 at 12.00 – 13.00 in the office of Faisal Altaf. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

GOOD LUCK!

Problem 1.

a. Explain what is meant by *recursive feasibility.* (2 p)

b. In the barrier method, an interior point method for constrained optimization, how are a) equality constraints, and b) inequality constraints, treated? (2 p)

c. You have been given the task to code an MPC algorithm, including a state estimator described by the recursions below with the initial/final conditions $\hat{x}(-1) = x_0$ and $x^0(T) = \hat{x}(T)$. Explain (i) what the overall aim of the algorithm is, and (ii) what is done in each of the 5 equations (one sentence per equation). (3 p)

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + L(k)\big(y(k) - C(A\hat{x}(k-1) + Bu(k-1))\big) \tag{1}$$

$$x^0(k) = \hat{x}(k) + P_m(k)A^T[AP_m(k)A^T + Q]^{-1}\big(x^0(k+1) - A\hat{x}(k)\big) \tag{2}$$

$$P_t(k) = AP_m(k-1)A^T + Q \tag{3}$$

$$L(k) = P_t(k)C^T[CP_t(k)C^T + R]^{-1} \tag{4}$$

$$P_m(k) = P_t(k) - P_t(k)C^T[CP_t(k)C^T + R]^{-1}CP_t(k) \tag{5}$$

d. Consider the following system with two inputs and two outputs:

$$A = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \\ 0.25 & 0 \\ 0 & 0.6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Show that the output setpoint $y_{sp} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ is a feasible steady state target. (2 p)

e. The terminal cost plays an important role when proving stability of a receding horizon control strategy, and this can be made very explicit in the *unconstrained* LQ case. Thus, show how an infinite-horizon LQ objective can be transformed into an equivalent finite-horizon objective — and show all important steps! (2 p)

Solution:

a. *Recursive feasibility is the property that solving the MPC optimization problem for an initial feasible state results in the next state being feasible as well. This results in a sequence of feasible (solvable) optimal control problems. All this holds if there are no uncertainties.*

b. *Equality constraints are respected during the search, using e.g. Newton's method. Inequality constraints are handled by adding a log barrier function to the objective function, adding a large penalty for (close to) infeasible points.*

c. *The overall aim is to implement a state estimator, that delivers the least-squares estimates based on a "batch" of data over an interval* $[0\ T]$ *(alternatively, the aim is to implement a moving horizon estimator with a data window of length* T*). The five equations perform the following steps: (1) time/measurement update of filter estimate; (2) reverse time update of smoothing estimate; (3) time update of covariance matrix; (4) calculation of filter gain; (5) measurement update of covariance matrix.*

d. *The steady-state target should fulfil the equation*

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix} \tag{6}$$

In this case it is possible to verify by hand without great difficulties that $x_s^T = \begin{bmatrix} 2.5, & -1.5, & 1.25, & -2.25 \end{bmatrix}$, $u_s^T = \begin{bmatrix} 2.5, & -1.5 \end{bmatrix}$ *is a solution for the given* y_{sp}.

e. *The optimal infinite-horizon objective can be re-written as*

$$\begin{aligned} V_\infty^0 &= \min_{u(0:\infty)} \sum_{i=0}^{\infty} (x^T(i)Qx(i) + u^T(i)Ru(i)) \\ &= \min_{u(0:N-1)} \Big(\sum_{i=0}^{N-1} (x^T(i)Qx(i) + u^T(i)Ru(i)) + \min_{u(N:\infty)} \sum_{i=N}^{\infty} (x^T(i)Qx(i) + u^T(i)Ru(i)) \Big) \\ &= \min_{u(0:N-1)} \Big(\sum_{i=0}^{N-1} (x^T(i)Qx(i) + u^T(i)Ru(i)) + x^T(N)Px(N) \Big) \end{aligned}$$

where the last step follows from the fact that the optimal cost-to-go is quadratic in the initial state with P *being the solution of the algebraic Riccati equation.*

Problem 2.

Consider a standard constrained LQ type MPC applied to a system, for which it is very important to avoid unnecessary oscillations in connection with (infrequent) changes of the usually constant setpoint. Therefore, to avoid oscillations in the closed-loop step response, we would like to add suitable constraints, specifically when setpoint changes are made.

a. Assuming that the setpoint is *increased* to a new constant value that is larger than the current output, show how affine inequalities in the decision variables can be used to demand that the (predicted) output increases *monotonically* as a response to the changed setpoint.
Hint: Assume that we choose to formulate the MPC algorithm with both state variables and control inputs as decision variables, and that we have access to full state information. (2 p)

b. What needs to be changed if instead the setpoint is *decreased*? Which is the implication for the implementation of the MPC algorithm? (2 p)

Solution:

a. The requirement on monotonic response to an increased *setpoint corresponds to enforcing the inequalities (k is the current sampling time)*

$$y(k+i) \leq y(k+i+1), \qquad i = 0, \ldots, N-1,$$

where future output predictions are expressed in terms of decision variables as $y(k+i) = Cx(k+i|k)$ for $i = 1, \ldots, N-1$ and the current output is given by $y(k) = Cx(k)$ according to assumptions. Collecting all inequalities in vector form, we get

$$\begin{bmatrix} -C & & & & \\ C & -C & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & C & -C \end{bmatrix} \begin{bmatrix} x(k+1|k) \\ x(k+2|k) \\ \vdots \\ \vdots \\ x(k+N|k) \end{bmatrix} \leq \begin{bmatrix} -Cx(k) \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

which is clearly an affine inequality in the decision variables.

b. If the setpoint is decreased, the inequality in (a) will be reversed. The implication is that the implemented MPC will have different inequalities, depending on the direction of the setpoint (and none of these if the setpoint remains unchanged). The resulting MPC controller will thus not be time-invariant as we are used to.

Problem 3.
Consider a first order system described by the model

$$x(k+1) = 0.5x(k) + u(k), \quad u(k) \geq 0$$

where it should be noticed that only non-negative control inputs are allowed. We want to construct an LQ based MPC for this system based on minimizing the 1-step ahead cost function

$$V_1(x(0), u(0)) = x^2(1) + u^2(0),$$

where as usual 'current time' k has been placed at the origin.

a. Determine the cost function $V_1(x, u)$ as a function of current state $x = x(0)$ and control candidate $u = u(0)$. Based on this, determine the control law resulting from the *unconstrained* LQ problem. (2 p)

b. Now assume that the control constraint is taken into consideration, i.e. we would like to minimize V_1 under the constraint

$$u \geq 0$$

Determine the control law resulting from the constrained MPC formulation. (2 p)

c. Show that the state converges to zero for the closed-loop system obtained with the constrained MPC. (1 p)

Solution:

a. *The cost function is*

$$V_1(x, u) = x^2(1) + u^2(0) = (0.5x + u)^2 + u^2 = 2(u + \frac{x}{4})^2 + \frac{1}{8}x^2$$

From this follows that the unconstrained control law, minimizing V_1, is given by

$$u = -\frac{x}{4}$$

b. *With the constraint on u, the minimizing control is not any longer allowed for positive x. However, for positive x, it follows from the expression*

$$V_1(x, u) = 2u^2 + xu + \frac{1}{4}x^2$$

that $V_1(x,u)$ is minimized by the choice $u = 0$. The constrained MPC control law is hence

$$u(x) = \begin{cases} -x/4 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

c. *From (b) it follows that the closed-loop system is described by either of two equations*

$$x(k+1) = \begin{cases} 0.25x(k) & x(0) < 0 \\ 0.5x(k) & x(0) \leq 0 \end{cases}$$

i.e the system is governed by the open-loop dynamics if the initial state is negative. In either case, the state converges to zero exponentially and the closed-loop system is stable.

Problem 4.

You want to apply an MPC based on constrained LQ to a plant described by a standard, continuous-time linear state space model (A, B, C). The problem you have is that you have concluded that you need a fast sampling time in the range of 1 ms, and the computations for the MPC takes almost 0,3 ms, i.e. a significant part of the sampling period.

a. Show how the continuous-time state-space model can be discretized in such a way that the next sampled state $x(k+1)$ can be expressed in terms of the current state $x(k)$ and two consecutive control inputs (assumed piecewise constant as usual). (2 p)

b. Based on the result in (a), give a discrete-time state-space model of the plant in standard form. (2 p)

Solution:

a. *With the computational delay $\tau = 0.3ms$, the control signal over the sampling interval is given by*

$$u(t) = \begin{cases} u(k-1), & kh \leq t < kh + \tau \\ u(k), & kh + \tau \leq t < (k+1)h \end{cases}$$

and the continuous-time state equation can be solved as

$$\begin{aligned}
x(k+1) &= e^{A(h-\tau)}x(kh+\tau) + \int_0^{h-\tau} e^{As}B\,ds \cdot u(k) \\
&= e^{A(h-\tau)}\left[e^{A\tau}x(kh) + \int_0^{\tau} e^{As}B\,ds \cdot u(k-1)\right] + \int_0^{h-\tau} e^{As}B\,ds \cdot u(k) \\
&= e^{Ah}x(kh) + e^{A(h-\tau)}\int_0^{\tau} e^{As}B\,ds \cdot u(k-1) + \int_0^{h-\tau} e^{As}B\,ds \cdot u(k) \\
&= A_d x(k) + B_1 u(k-1) + B_2 u(k)
\end{aligned}$$

b. *Introducing the augmented state vector*

$$\xi(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}$$

the following standard state-space model is obtained:

$$\xi(k+1) = \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_d & B_1 \\ 0 & 0 \end{bmatrix} \xi(k) + \begin{bmatrix} B_2 \\ I \end{bmatrix} u(k)$$

Problem 5.
A sampled data model for a DC motor is given by

$$\begin{aligned}
x(k+1) = Ax(k) + Bu(k) &= \begin{bmatrix} 1 & 0.14 \\ 0 & 0.86 \end{bmatrix} x(k) + \begin{bmatrix} 0.21 \\ 2.79 \end{bmatrix} u(k) \\
y(k) = Cx(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)
\end{aligned}$$

We will investigate an MPC for this system, based on the minimisation of the quadratic criterion

$$V(x, u(0), u(1)) = \sum_{i=1}^{4} \hat{y}^2(i)$$

with respect to the control signals $u(0)$ and $u(1)$, where we have (as usual) placed the time origin at the current sampling time. It can be seen that the prediction horizon is 4 and the control horizon is 2.

a. The predicted outputs over the prediction horizon can be computed from the equation

$$\begin{bmatrix} \hat{y}(1) \\ \hat{y}(2) \\ \hat{y}(3) \\ \hat{y}(4) \end{bmatrix} = \Psi x + \Phi \mathcal{U}$$

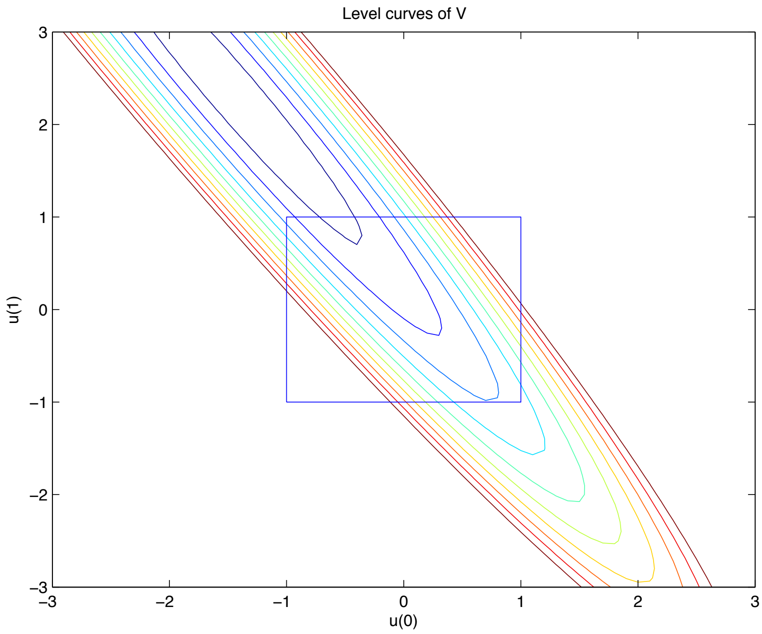
where x is the current state and $\mathcal{U} = (u(0)\ u(1))^T$. Using the assumption that the control signal is equal to 0 beyond the control horizon (i.e. in contrast to the usual assumption that the control signal stays constant beyond the control horizon), give expressions for Φ and Ψ in terms of the system matrices A, B and C. Do not bother about numerical values at this stage! (2 p)

b. Calculate the *unconstrained* optimal control law *numerically*. (2 p)
Hint: Here are some numerical expressions that *may be* useful:

$$\Psi = \begin{bmatrix} 1.00 & 0.14 \\ 1.00 & 0.26 \\ 1.00 & 0.36 \\ 1.00 & 0.45 \end{bmatrix} \qquad \Phi = \begin{bmatrix} 0.21 & 0 \\ 0.60 & 0.21 \\ 0.94 & 0.60 \\ 1.22 & 0.94 \end{bmatrix}$$

$$\Phi^T\Phi = \begin{bmatrix} 2.78 & 1.84 \\ 1.84 & 1.29 \end{bmatrix} \qquad \Phi^T\Psi = \begin{bmatrix} 2.98 & 1.08 \\ 1.75 & 0.70 \end{bmatrix}$$

c. We are now interested in the case with *control constraints.* In this case, the minimization of V should be carried out with the constraints $|u(0)| \leq 1$ and $|u(1)| \leq 1$. In the figure below, the level curves of V are shown for a specific value of the state x, namely $x = (1 \ \ -3)^T$. The box indicates the control constraints.
Which value of the control signal is computed by the constrained MPC? How does this compare with the saturated ($-1 \leq u \leq 1$) control signal from the unconstrained control law? (2 p)



Solution:

a. *The matrices are found by iterating the state equations (see chapter 5 of the Lecture notes) and using the fact that* $u(2) = u(3) = 0$*:*

$$\Psi = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} \qquad \Phi = \begin{bmatrix} CB & 0 \\ CAB & CB \\ CA^2B & CAB \\ CA^3B & CA^2B \end{bmatrix}$$

b. *Minimizing*

$$V = (\Psi x + \Phi \mathcal{U})^T(\Psi x + \Phi \mathcal{U}) = \mathcal{U}^T\Phi^T\Phi\mathcal{U} + 2x^T\Psi^T\Phi\mathcal{U} + x^T\Psi^T\Psi x$$

gives the unconstrained minimum

$$\mathcal{U} = -(\Phi^T\Phi)^{-1}\Phi^T\Psi x \approx \begin{bmatrix} -3.11 & -0.52 \\ 3.08 & 0.21 \end{bmatrix} x$$

Only the first control signal is used and gives the control law

$$u = \begin{bmatrix} -3.11 & -0.52 \end{bmatrix} x$$

c. *The unconstrained control law gives*

$$u = \begin{bmatrix} -3.11 & -0.52 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = -1.55,$$

which after saturation gives $u = -1$.
However, from the diagram of the level curves, it is seen that the minimum value inside the box is obtained at the upper boundary for $u(1) = 1$ *and approximately* $u(0) = -0.6$. *The latter is the control signal delivered by the constrained MPC.*

THE END!