CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Exam 2014-03-13

08:30 – 12:30

Teachers: Bo Egardt, tel 3721. Emil Klintberg (tel 2215) will visit twice during the exam.

The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 sheet with your own notes.
- Mathematics Handbook (Beta).

Note: Solutions should be given in **English**! They may be short, but should always be **clear, readable and well motivated**!

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5).
Review of the grading is offered on April 1 at 12.00 – 13.00. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

GOOD LUCK!

Problem 1.

a. What is the reason that the DMC scheme can be used for stable plants only? (2 p)

b. It has been demonstrated during the course that the moving horizon estimate can be computed using the same recursions as used in the Kalman filter. Explain why the two estimates are, in general, yet different. (2 p)

c. Explain what is meant by *recursive feasibility.* (2 p)

d. What is meant by a soft constraint? Explain in words and give an example in terms of a couple of equations. (2 p)

e. What is meant by explicit MPC? What are its pros and cons (one each is sufficient)? (2 p)

Solution:

a. The state observer used in the DMC scheme is partly (the part corresponding to the original state) open-loop.

b. The reason that the estimates differ is that the recursions for the MHE have to be initialized at the beginning of the sliding window, i.e. have to be re-run at every sampling instant, whereas the Kalman filter is initialized only once.

c. Recursive feasibility is the property that solving the MPC optimization problem for an initial feasible state results in the next state being feasible as well. This results in a sequence of feasible (solvable) finite horizon optimal control problems. All this holds if there are no uncertainties.

d. A soft constraint can be violated if needed, but adds cost to the objective function. One way to do this is to introduce a slack variable ε *according to*

$$\begin{aligned} \min_{\mathcal{U}} \quad & V_N(\mathcal{U}) + \rho\|\varepsilon\|^2 \\ \textit{subject to} \quad & F\,\mathcal{U} + G\,\mathcal{X} \leq e + \varepsilon \\ & \varepsilon \geq 0 \end{aligned}$$

e. *In explicit MPC, the multi-parametric optimization problem is solved off-line, resulting in a piece-wise affine control law that is stored in memory. The on-line computation is thereby limited to looking up the controller parameters corresponding to the current state, and then to compute the control signal. The advantage is that most of the computations can be done off-line. The disadvantages are that storage requirements can be very large, and that the off-line computations can be prohibitive.*

Problem 2.
Consider a first order system described by the model

$$x(k+1) = 0.5x(k) + u(k), \quad u(k) \geq 0$$

where it should be noticed that only non-negative control inputs are allowed. We want to construct an LQ based MPC for this system based on minimizing the 1-step ahead cost function

$$V_1(x(0), u(0)) = x^2(1) + u^2(0),$$

where as usual 'current time' k has been placed at the origin.

a. Determine the cost function $V_1(x, u)$ as a function of current state $x = x(0)$ and control candidate $u = u(0)$. Based on this, determine the control law resulting from the *unconstrained* LQ problem. (2 p)

b. Now assume that the control constraint is taken into consideration, i.e. we would like to minimize V_1 under the constraint

$$u \geq 0$$

Determine the control law resulting from the constrained MPC formulation. (2 p)

c. Show that the state converges to zero for the closed-loop system obtained with the constrained MPC. (1 p)

Solution:

a. The cost function is

$$V_1(x, u) = x^2(1) + u^2(0) = (0.5x + u)^2 + u^2 = 2(u + \frac{x}{4})^2 + \frac{1}{8}x^2$$

From this follows that the unconstrained control law, minimizing V_1, is given by

$$u = -\frac{x}{4}$$

b. With the constraint on u, the minimizing control is not any longer allowed for positive x. However, for positive x, it follows from the expression

$$V_1(x, u) = 2u^2 + xu + \frac{1}{4}x^2$$

that $V_1(x, u)$ is minimized by the choice $u = 0$. The constrained MPC control law is hence

$$u(x) = \begin{cases} -x/4 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

c. *From (b) it follows that the closed-loop system is described by either of two equations*

$$x(k+1) = \begin{cases} 0.25x(k) & x(0) < 0 \\ 0.5x(k) & x(0) \leq 0 \end{cases}$$

i.e the system is governed by the open-loop dynamics if the initial state is negative. In either case, the state converges to zero exponentially and the closed-loop system is stable.

Problem 3.

Predictions of states and/or outputs are important for MPC. In this problem, we will investigate how predictions can be constructed from transfer function or difference equation models including noise terms. Consider the first order system (where, for simplicity, the control input has been omitted) given by the difference equation

$$y(k) + ay(k-1) = e(k) + ce(k-1)$$

where the white noise e has standard deviation σ. Assume that $|c| < 1$.

a. Determine a state space model for the system.
Hint: Start by identifying the direct term D of the state-space model (A, B, C, D). (2 p)

b. Determine the steady state Kalman filter for the system, and put it in transfer function form. Note that in this case, the process noise and the measurement noise will be correlated. Denoting the cross correlation S, the time-varying Kalman filter equations are modified slightly; in prediction form, they are given by (note that $B = 0$ in our simplified problem)

$$\begin{aligned} \hat{x}(k+1|k) &= A\hat{x}(k|k-1) + Bu(k) \\ &\quad + (AP(k)C^T + S)[CP(k)C^T + R]^{-1}(y(k) - C\hat{x}(k|k-1)) \\ P(k+1) &= AP(k)A^T + Q \\ &\quad - (AP(k)C^T + S)[CP(k)C^T + R]^{-1}(CP(k)A^T + S) \end{aligned}$$

(2 p)

c. Determine an expression for the steady-state, one-step ahead prediction of the output. (2 p)

Solution:

a. Using the hint, write the system model as

$$y(k) = \frac{1 + cz^{-1}}{1 + az^{-1}} e(k) = \frac{c - a}{1 + az^{-1}} e(k-1) + e(k)$$

from which we see that $D = 1$. *It seems natural to choose* $x(k) = \frac{1}{1+az^{-1}} e(k-1)$, *which gives the state space model*

$$\begin{aligned} x(k+1) &= -ax(k) + e(k) \\ y(k) &= (c-a)x(k) + e(k) \end{aligned}$$

b. In steady state, the P *equation for the model above becomes*

$$P = a^2 P + \sigma^2 - \frac{\left(\sigma^2 - aP(c-a)\right)^2}{(c-a)^2 P + \sigma^2}$$

with the solution $P = 0$ *(found easily by first testing the case* $c = 0$*). The steady state Kalman gain is then* $L = S/R = 1$ *and the filter becomes*

$$\hat{x}(k+1|k) = -a\hat{x}(k|k-1) + 1 \cdot (y(k) - (c-a)\hat{x}(k|k-1)) = -c\hat{x}(k|k-1) + y(k)$$

which can be written in transfer function form

$$\hat{x}(k+1|k) = \frac{1}{1 + cz^{-1}} y(k)$$

c. Since $e(k)$ *is white noise, we have*

$$\hat{y}(k+1|k) = (c-a)\hat{x}(k+1|k) = \frac{c-a}{1+cz^{-1}} y(k)$$

Problem 4.

Consider the augmented model

$$\begin{bmatrix} x \\ d \end{bmatrix}^+ = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

where the constant but unknown disturbance d affects both state and output equations through the matrices B_d and C_d. In this problem, we will investigate a special case (familiar from Assignment 2) with the following assumptions:

- $C = I$ (the identity matrix)
- $B_d = 0$
- C_d is of full column rank

In order to estimate the unknown disturbance, the observability of the augmented system is of interest.

a. Show that the augmented system is always observable if A has no eigenvalue equal to 1. (2 p)

b. Show that the augmented system is not observable if 1 is an eigenvalue of A and d has as many components as the number of outputs (and states). (2 p)

Hint: The pair (C, A) is observable if and only if any of the following condition holds (n is the number of state variables):

- The matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank n
- The matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has rank n for all $\lambda \in \mathbb{C}$

Solution:

a. *Suppose the augmented system is not observable, i.e. there is a non-zero vector* $[v_1^T\ v_2^T]^T$ *such that, for some* λ,

$$\begin{bmatrix} \lambda I - A & 0 \\ 0 & (\lambda - 1)I \\ I & C_d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

First, assume $\lambda \neq 1$. *It then follows from the second block row that* $v_2 = 0$. *From the third block-row it then follows that* $v_1 = 0$. *Hence, we must have* $\lambda = 1$. *Since 1 is not an eigenvalue of* A, *it follows from the first block-row that* $v_1 = 0$. *From the third block-row we then have* $C_d v_2 = 0$, *which implies* $v_2 = 0$ *since* C_d *has full column rank. We conclude that the system is observable.*

b. *When 1 is an eigenvalue of* A, *there is a non-zero vector* v_1 *such that* $(I - A)v_1 = 0$. *Since* C_d *is now square and invertible (full rank), we can choose* $v_2 = -C_d^{-1} v_1$. *With these choices, the observability test shows that the augmented system is not observable.*

Problem 5.
Consider the quadratic program

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2} x^T Q x + p^T x, \quad Q > 0 \\ \text{subject to} \quad & Ax = b, \quad A \text{ has full row rank} \end{aligned}$$

a. Determine the Lagrange dual function to this problem. (2 p)

b. Determine the dual optimal solution. (2 p)

c. By combining the results from a) and b), show that the optimal x and ν can be computed as the solution of a system of linear equations (which in fact expresses the KKT conditions for the problem). (1 p)

Solution:

a. *The Lagrange dual function is given by*

$$q(\nu) = \inf_x L(x, \nu) = \inf_x \frac{1}{2} x^T Q x + p^T x + \nu^T (Ax - b)$$

The infimum can be found by differentiating L *with respect to* x, *giving* $Qx^0 = -(p + A^T \nu)$, *so that*

$$q(\nu) = -\frac{1}{2}(p + A^T \nu)^T Q^{-1} (p + A^T \nu) - b^T \nu$$

b. The dual optimal solution is determined by maximizing the dual function w.r.t. ν, which can be done by differentiating q and setting the derivative equal to zero, giving

$$-AQ^{-1}(p + A^T\nu^0) = b$$

and hence

$$AQ^{-1}A^T\nu^0 = -(AQ^{-1}p + b)$$

c. Combining the two expressions above for x^0 and ν^0 gives

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ \nu^0 \end{bmatrix} = \begin{bmatrix} -p \\ b \end{bmatrix}$$

which is the system of linear equations defining the optimal solution (and which forms the KKT conditions).