CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Exam 2013-03-12

V 08.30 – 12.30

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The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 sheet with your own notes.
- Mathematics Handbook (Beta).

Note: Solutions should be given in **English**! They may be short, but should always be **clear, readable and well motivated**!

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5).
Review of the grading is offered on April 2 at 12.00 – 13.00. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

GOOD LUCK!

Problem 1.

a. The MPC algorithm studied during the course is based on i) a linear model; ii) quadratic objective, and iii) affine constraints. Explain in what way these limitations contribute to make the MPC optimization problem *convex*. (2 p)

b. The standard plant model used in the course is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= C_y x(k) \\ z(k) &= C_z x(k) \end{aligned}$$

Show how this model can be transformed into a model that instead uses *control moves* Δu as input by introducing the new state vector

$$\xi(k) = \begin{bmatrix} \Delta x(k) \\ y(k-1) \\ z(k-1) \end{bmatrix}$$

(2 p)

c. The optimization problem of an MPC algorithm may turn out to be *infeasible*. Explain what is meant by infeasibility and explain what *specific* part of the MPC that is the cause of this. (2 p)

d. What is meant by the term *explicit MPC*? Which property of the MPC studied in the course is exploited in explicit MPC? (2 p)

e. Explain the principal workings of an *active set method* for constrained optimization. (2 p)

Solution:

a. *The* objective is convex*, since a quadratic function with positive semidefinite Hessian is convex. The linear model implies that state predictions and future controls are related via* affine equality constraints*. Affine constraints on input and state give affine, and therefore* convex inequality constraints*. Hence, the conditions for a convex optimization problem are fulfilled.*

b. *The new model is*

$$\xi(k+1) = \begin{bmatrix} A & 0 & 0 \\ C_y & I & 0 \\ C_z & 0 & I \end{bmatrix} \xi(k) + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \Delta u(k)$$
$$y(k) = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \xi(k)$$
$$z(k) = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \xi(k)$$

c. *The optimization problem is infeasible if there is no value of the optimization variable that fulfills the constraints. Infeasibility of an MPC algorithm is caused by state or output constraints (*not *input constraints!).*

d. *(Problem excluded).*

e. *An active set method solves a sequence of problems with* equality *constraints only. The sequence is obtained by treating inequality constraints either as* active, *in which case the constraint is included as an equality constraint, or as* passive, *in which case it is disregarded. In principle, all combinations of active/passive inequality constraints may have to be investigated.*

Problem 2.
Consider the system described by

$$y(k+1) = u(k) + bu(k-1)$$

You will investigate an MPC controller for this system that is based on minimization of the objective

$$V_2(u(k|k), u(k+1|k)) = (\hat{y}(k+1|k) - r(k))^2 + (\hat{y}(k+2|k) - r(k))^2$$

where $r(k)$ is the setpoint or reference signal, and both the prediction horizon and the control horizon are equal to 2.

a. Solve the optimization problem and give an expression for the control law. (3 p)

b. Determine under what conditions the closed-loop system is stable. (2 p)

Solution:

a. *The predictions are given by*

$$\hat{y}(k+1|k) = u(k|k) + bu(k-1)$$
$$\hat{y}(k+2|k) = u(k+1|k) + bu(k|k)$$

To find the minimizing control, differentiate V_2 *w.r.t.* $\hat{u}(k|k)$ *and* $\hat{u}(k+1|k)$*, respectively, and set the result equal to zero:*

$$(u(k|k) + bu(k-1) - r(k)) \cdot 1 + (u(k+1|k) + bu(k|k) - r(k)) \cdot b = 0$$
$$(u(k|k) + bu(k-1) - r(k)) \cdot 0 + (u(k+1|k) + bu(k|k) - r(k)) \cdot 1 = 0$$

which gives the minimizing control (we focus on $u(k|k) = u(k)$*):*

$$u(k) = -bu(k-1) + r(k)$$

b. *The control law in z-transformed version is*

$$U(z) = \frac{1}{1 + bz^{-1}} R(z)$$

which implies that the plant zero in $-b$ *will be cancelled by the controller pole in the same location. The cancellation implies that the closed-loop input-output relation becomes* $y(k+1) = r(k)$*. However, the cancelled pole determines the stability, so that the condition for closed-loop stability is* $|b| < 1$*, which means that the process must be minimum phase.*

Problem 3.

The task in this problem is to give a description of the basic program flow in an MPC algorithm of the type considered during the course. The following parts of the algorithm should be described:

- State estimation (described by equations)
- Steady state target calculation (described by equations)
- Calculation of the control signal (represented by a subroutine call)

Please be very clear about in which order these parts are executed, and which are the input and output data for each part!

The following should be observed:

- Assume that the computation time is short compared to the sampling interval.

- If current time is k, then the latest available plant output is $y(k)$, and the algorithm shall produce the control $u(k)$.
- The state estimator shall make use of the latest available information.
- You should describe how offsets and deviation variables are handled.

Hint: State estimation can for example be based on the Kalman filter (equations for the Kalman gain $L(k)$ need not be given):

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + L(k)[y(k) - C\hat{x}(k|k-1)] \\ \hat{x}(k+1|k) &= A\hat{x}(k|k) + Bu(k) \end{aligned}$$

Solution: *If a disturbance d is included, an augmented model with state $x_e = (x,\, d)$ is used. The matrices of the augmented model are here denoted $\mathcal{A}, \mathcal{B}, \mathcal{C}$.*

a. *Perform observer measurement update:*

$$\hat{x}_e(k|k) = \hat{x}_e(k|k-1) + L(k)[y(k) - \mathcal{C}\hat{x}_e(k|k-1)]$$

b. *Update steady state target (here with disturbance estimate included):*

$$\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s(k) \\ u_s(k) \end{bmatrix} = \begin{bmatrix} B_d\hat{d}(k|k-1) \\ z_{sp} - HC_d\hat{d}(k|k-1) \end{bmatrix}$$

c. *Calculate the control signal:*

   (i) $\tilde{x}(k) = \hat{x}(k|k) - x_s(k)$

   (ii) $\tilde{u}(k) = QP(\tilde{x}(k))$

   (iii) $u(k) = \tilde{u}(k) + u_s(k)$

d. *Perform observer time update (to prepare for next sampling instant):*

$$\hat{x}(k+1|k) = \mathcal{A}\hat{x}(k|k) + \mathcal{B}u(k)$$

(5 p)

Problem 4.
We shall investigate a variation of a simple MPC studied during the course. The plant is an integrator process

$$x^+ = x + u$$

and an MPC with horizon $N = 2$ and control constraint $-1 \leq u \leq 1$ was shown in the lectures to give a control law that could be described as a saturated linear feedback.
The expression for the objective can be written

$$V_N(x, \mathcal{U}) = \mathcal{U}^T H \mathcal{U} + 2 \cdot [2x \;\; x]\mathcal{U} + 3x^2,$$

where $x = x(0)$, $\mathcal{U} = [u(0) \;\; u(1)]^T$ and $H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

In contrast to the problem investigated earlier, and motivated by the fact that only the first control $u(0)$ will actually be used, the control constraint will now be put on *the first control action only*, implying that there are only two constraints, given by the inequalities

$$-1 \leq u(0) \leq 1$$

a. Use the KKT conditions to derive the control law for this modified problem. (5 p)

Hint: The KKT conditions for this case (i.e. no equality constraints) are given by (note that in the following, x is a general notation for the vector of decision variables) are:

(i) Primal constraints: $g_i(x) \leq 0$, $i = 1, \ldots, m$

(ii) Dual constraints: $\lambda_i \geq 0$, $i = 1, \ldots, m$

(iii) Complementary slackness: $\lambda_i g_i(x) = 0$, $i = 1, \ldots, m$

(iv) Gradient of the Lagrangian equal to zero:

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

Solution: *The KKT conditions become*

$$\begin{gathered} g_1 = u(0) - 1 \leq 0 \\ g_2 = -u(0) - 1 \leq 0 \\ \lambda \geq 0 \\ \lambda_i g_i = 0, \quad i = 1, 2 \\ 2\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \mathcal{U} + 2\begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} \lambda_1 - \lambda_2 \\ 0 \end{bmatrix} = 0 \end{gathered}$$

Note that λ_1, λ_2 can not be nonzero simultaneously. There are therefore 3 cases to consider:

(i) $\lambda_1 = \lambda_2 = 0$ *(no active constraints, i.e.* $-1 \leq u(0) \leq 1$*):*

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \mathcal{U} + \begin{bmatrix} 2x \\ x \end{bmatrix} = 0 \Rightarrow \mathcal{U} = \begin{bmatrix} -3/5 \\ -1/5 \end{bmatrix} x$$

Since $-1 \leq u(0) \leq 1$*, it follows that this case is valid in the region* $-5/3 \leq x \leq 5/3$*.*

(ii) $\lambda_1 > 0, \lambda_2 = 0$ *(i.e.* $u(0) = 1$*):*

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ u(1) \end{bmatrix} + \begin{bmatrix} 2x \\ x \end{bmatrix} = \begin{bmatrix} -\lambda_1/2 \\ 0 \end{bmatrix} \Rightarrow u(1) = -\frac{1+x}{2}, \; x < -5/3$$

(iii) $\lambda_1 = 0, \lambda_2 > 0$ *(i.e.* $u(0) = -1$*):*

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ u(1) \end{bmatrix} + \begin{bmatrix} 2x \\ x \end{bmatrix} = \begin{bmatrix} \lambda_2/2 \\ 0 \end{bmatrix} \Rightarrow u(1) = \frac{1-x}{2}, \; x > 5/3$$

The control law is thus given by

$$u(x) = \begin{cases} 1, & x < -5/3 \\ -3/5 \cdot x, & -5/3 \leq x \leq 5/3 \\ -1, & x > 5/3 \end{cases}$$

Problem 5.

Stability of the closed-loop system when a model predictive controller is applied to the process

$$x^+ = f(x, u)$$

can in some cases be proved from the following basic property:

$$\min_{u \in \mathbb{U}} \{V_f(f(x,u)) + l(x,u) \,|\, f(x,u) \in \mathbb{X}_f\} \leq V_f(x), \quad \forall x \in \mathbb{X}_f$$

Here, l is the stage cost, V_f is the terminal cost, and $\mathbb{X}_f$ is the terminal constraint set.

Show how this property can be fulfilled for the standard MPC algorithm studied in the course (i.e. linear plant, quadratic cost and affine constraints) by appropriate choices of V_f and $\mathbb{X}_f$.

Hint 1: You do not have to give an explicit expression for $\mathbb{X}_f$, but its properties should be specified.

Hint 2: The infinite horizon LQ controller is given by the equations

$$
\begin{aligned}
K &= -(B^T PB + R)^{-1} B^T PA \\
P &= Q + A^T PA - A^T PB(B^T PB + R)^{-1} B^T PA
\end{aligned}
$$

(5 p)

Solution: *For our standard case, we have the system description*

$$x^+ = Ax + Bu$$

With the choice $V_f = x^T Px$, where P is the solution to the algebraic Riccati equation, we can evaluate V_f at the next state:

$$
\begin{aligned}
V_f(x^+) &= \big((A + BK)x\big)^T P\big((A + BK)x\big) \\
&= x^T A^T PAx + x^T K^T B^T PBKx + 2x^T K^T B^T PAx
\end{aligned}
$$

Use the Riccati equation and the expression for K to rewrite the first term of the RHS:

$$
\begin{aligned}
x^T A^T PAx &= x^T Px - x^T Qx + x^T A^T PB(B^T PB + R)^{-1} B^T PAx \\
&= x^T Px - x^T Qx - x^T K^T B^T PAx
\end{aligned}
$$

Combining the results, we get

$$
\begin{aligned}
V_f(x^+) &= x^T Px - x^T Qx + x^T K^T (B^T PBK + B^T PA)x \\
&= V_f(x) - x^T Qx - x^T K^T RKx \\
&= V_f(x) - x^T Qx - u^T Ru \\
&= V_f(x) - l(x, u); \quad u = Kx
\end{aligned}
$$

which shows the decay property given in the problem, provided the infinite horizon LQ controller is used to give u and provided that constraints are not active in the set $\mathbb{X}_f$. *The latter can be guaranteed by selecting $\mathbb{X}_f$ to be a set (containing the origin) where constraints are not active, and with the property that the LQ controller keeps the state inside this set, once inside.*